SchoolNova Math Club: Vectors

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1. (a) Consider a vector $\mathbf{v}$ in $\mathbb{R}^2$ (i.e., 2-dimensional space), prove that its x-coordinate is equal to $v\cos\theta$, where $v = |\mathbf{v}|$ is the magnitude of $\mathbf{v}$ and θ is the angle the vector makes with the x-axis.

   (b) Similarly we can create an *orthogonal projection* of a vector $\mathbf{v}$ onto any unit vector $\mathbf{u}$, which is the base of the right triangle formed using $\mathbf{v}$ as a hypotenuse and legs parallel and perpendicular to $\mathbf{u}$. Prove that the length of the orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$ is equal to $v\cos\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{u}$. Let this quantity be denoted as $\mathbf{v}\cdot\mathbf{u}$, called the *dot product* of $\mathbf{v}$ with $\mathbf{u}$.

   (c) Prove that such a projection has a distributive property - namely, given unit vector $\mathbf{u}$ and any vectors $\mathbf{w}$, $\mathbf{v}$, that $(\mathbf{w}+\mathbf{v})\cdot\mathbf{u} = \mathbf{w}\cdot\mathbf{u} + \mathbf{v}/cdot\mathbf{u}$.

   (d) If two vectors are orthogonal (perpendicular), we say their orthogonal projection is 0 (trivially, neither vector has a nonzero component in the direction of the other vector). Notice also that if two vectors $\mathbf{v}$, $\mathbf{u}$ are parallel, then $\mathbf{v}\cdot\mathbf{u} = |\mathbf{v}||\mathbf{u}|$ (in this case since $\mathbf{u} = 1$, then this is just the magnitude of $\mathbf{v}$). Use this and the above distributive property to prove that $v\cos\theta = v_x u_x + v_y u_y$, where v_x is the x-component of $\mathbf{v}$, u_x is the x-component of $\mathbf{u}$, etc.

   (e) Finally, deduce that for any vectors $\mathbf{v}$, $\mathbf{w}$, $v_x w_x + v_y w_y = |\mathbf{v}||\mathbf{w}|\cos\theta$.

2. Consider a point p and a line l in $\mathbb{R}^2$, and let v be any point on l. We say a vector u is orthogonal to l if the line connecting v to $v+u$ is perpendicular to l. A vector is called a unit vector if it has magnitude (length) 1. Prove that there two unit vectors orthogonal to l. Then, prove that the minimum possible length of a line segment connecting p to any point on m is equal to $(p-v)\cdot u$, interpreting p, v as vectors and $\cdot$ is the dot product of vectors (see problem 1 for properties about the dot product).

3. In a triangle ABC the points D, E, F trisect the sides so that $BC = 3BD$, $CA = 3CE$ and $AB = 3AF$. Show that the triangles ABC and DEF have the same centroid.

4. Equilateral triangles are erected externally on the sides of a triangle A, B, C. Prove that the centroids of these triangles form an equilateral triangle.

5. Given a point P and vertices $A_1, A_2, ..., A_n$ of a regular n-gon inscribed in a circle of radius 1, prove that $PA_1^2 + PA_2^2 + \cdots + PA_n^2$ depends only on the distance of P from the center of the circle.

6. Let z_1, z_2, z_3 be complex numbers, interpretted as vectors in $\mathbb{R}^2$. Prove that these numbers are the vertices of an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

7. (a) Ash has a system of magic ketchup boxes connected by pipes. Each box has a blue half and a green half, which are separated, so that they are distinct compartments within the box. Ash has an infinite collection of magic pipes of four colors: green, blue, cyan, and yellow; each pipe has a power level, denoted by some real number.
   Given two boxes, Ash can transfer the ketchup in the first box to the second box by connecting pipes: let the first box's green and blue halves be denoted G1, B1, and the second box's halves be G2, B2. Connecting a green pipe between the boxes will copy p times the amount of ketchup in G1 into G2, where p is the power level of the pipe (negative ketchup is allowed). A blue pipe of power p will copy p times the content of B1 into B2; similarly, cyan pipes copy G1 contents into B2, and yellow pipes copy B1 contents into G2. Using combinations of such pipes, Ash can create desired amounts of ketchup in G2, B2 using the contents of G1, B1.

A set of four pipes, one of each color, is called a *charmander*. Given any two boxes, Ash can connect the first to the second using any charmander, then press a button on the charmander, activating the pipes and completing the transfer of ketchup. Charmanders will be denoted as sets of four numbers, for example (4,2,-1,0), indicating the powers of the green, blue, cyan, and yellow pipes, respectively.

(b) If Ash has three boxes with some ketchup in box 1, Ash can connect box 1 to box 2 and box 2 to box 3 to transfer ketchup from box 1 to box 3 using two charmanders. Notice that switching the order of the charmanders may change the final result in box 3.
Let boxes 2, 3 be empty. Given box 1 with contents (2,3), indicating two units of ketchup in its green half and three units of ketchup in its blue half, find an example of a pair of charmanders which give different results in box 3 if used in different orders.

(c) Let box 1 have contents (2,3), and box 2 be empty. Design a charmander that switches the green and blue halves, so that box 2 will end up with contents (3,2). Design a second charmander that doubles the content of each half, so that box 2 would end up with contents (4,6). Call these distinct charmanders c1, c2.
If three empty boxes are connected in series (meaning, a first box is connected to the second, then the second is connected to the third) using c1 and c2, and Ash puts some ketchup in the first box and then activates both c1 and c2, will the order of c1, c2 make a difference? Can you prove or disprove that, given any starting amount of ketchup, the end result in box 3 will be the same if c1, c2 are swapped?

(d) A pair of charmanders whose combined effect (in series) does not depend on the order is called commutative, and we say that one *commutes* with the other. Is there any charmander that commutes with *every* other charmander? (Hint: try (1,1,0,0).)

(e) If a charmander commutes with every other charmander, is it special in any particular way? Can you determine all such charmanders and what their numbers would be?

(f) A charmander is called a 2-cycle if it reverses its own action - meaning, if I connect box 1 to box 2 to box 3, initially empty, using two copies of the same charmander and then put some ketchup in box 1 and activate them, the result in box 3 will be exactly the same as the starting amount in box 1 (both the green and the blue half must be exactly the same as in box 1, no matter what those starting amounts are). Is (1,1,0,0) a 2-cycle? How about (-1,-1,0,0)?

(g) Similarly 3-cycles, 4-cycles, etc. can be defined, where an n-cycle connects a series of n+1 boxes. Prove that (-1,1,0,0) is not a 2-cycle. Is it a 3-cycle or a 4-cycle, or neither?

(h) Do 3-cycles exist? What if we allow complex number amounts of ketchup and pipes with complex number powers?

8. (a) Continuing from above, we define *reverse charmanders* as follows: a charmander c2 is the reverse of c1 if it undoes the action of c1, meaning if I connect them in series (c1 first, then c2), then activate both, the end result of ketchup will be exactly the same as the initial amounts, no matter what the initial amounts are. Prove that (1,1,-1,0) is the reverse of (1,1,1,0).

(b) If c2 is the reverse of c1, must c1 also be the reverse of c2? Can you provide proof or counterexample?

(c) Give an example of a charmander that has no reverse.

(d) Now define the *firepower* of a charmander as the product of its green and blue powers minus the product of its cyan and yellow powers: for example, (2,3,1,1) has firepower 5, and (-1,0,2,2) has firepower -4. Prove that 2-cycles have firepower 1.

(e) Prove that any n-cycle has firepower 1.

(f) If we know a charmander's firepower, can we deduce the firepower of its reverse, if it exists?

(g) If I put (1,2) in box 1 and (0,0) in box 2 and connect them with (1,1/2,-1,-1/2), what will the end result be in box 2? Such a result is called an *empty result*.

(h) Prove that if a charmander and some starting ketchup amount produces an empty result, then the charmander must have zero firepower. Is the converse true - if a charmander has zero firepower, must be there be some starting amount that produces an empty result?

(i) Prove that a charmander has a reverse if and only if it has nonzero firepower (hint: is an empty result reversible?).