MATH CLUB: INVARIANTS

MAR 4, 2018

In many problems, the problem is greatly simplified if you notice that there is some quantity which doesn't change when you do all operations allowed by the problem. Such a quantity is called an invariant. This is commonly used to prove that some configuration is impossible to achieve: if you find an invariant such that the value of this invariant for two configurations is different, then it is impossible to get one of these configurations from the other by using operations which do not change the invariant.

1. You have the standard 8×8 chessboard. You are allowed to switch the color of all the squares in one row or one column. Is it possible to achieve, by repeating such operations many times, that all but one squares are black?

2. The circle is divided into 6 sectors. In each of the sectors there is one coin. You are allowed to move exactly two coins in opposite directions: one to the next sector clockwise, the other tow the next sector counterclockwise. Is it possible to move all coins into the same sector by repeating this operation?

3. The following sequence of numbers is written on the blackboard:

$$1, 1/2, 1/3, \dots, 1/2018$$

Every second, a student chooses two numbers on the board, a and b, erases them and writes instead a single number $a+b+ab$. After 2017 such operations, there will be a single number on the blackboard.

What is this number?

4. We have an $n \times n$ board , divided by horizontal and vertical lines into 1×1 squares. Each of these 1×1 squares is colored either black or white, and all squares along the boundary of the board are black.

Let us call an edge between two squares bicolored if it separates a black cell from a white cell. Prove that the total number of bicolored edges is even.

5. A 100×100 yard field of wheat is divided into 1 yd $\times$ 1 yd squares. Initially, 9 of these squares were infected by some crop disease. The disease spreads as follows: for every square, if in the given year at least 2 of its 4 neighbors were infected, then next year the infection spreads to this square. (The squares that were infected stay infected forever). Prove that the disease will never spread to the whole field.

***6.** Let P be a polygon drawn on square ruled paper, so that all vertices of the polygon are the grid points. For such a polygon P, let $S(P)$ be a number defined as follows:

$$S(P) = \#(\text{grid points inside } P) + \frac{1}{2}\#(\text{grid points on the boundary of } P) - 1.$$

(a) Prove that if we cut a polygon by a segment into two polygons P_1, P_2, then $S(P) = S(P_1) + S(P_2)$. [The endpoints of the segment must be grid points.]

(b) Prove that for a rectangle whose sides follow the grid lines, we have $S(P) = $ area of$P$.

(c) Prove that the same is true for any triangle.

(d) Prove that this is true for any polygon.

(This result is called *Pick's formula).*