# MATH CLUB: RECURRENT SEQUENCES

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In many problems, a sequnce is defined using a recurrence relation, i.e. the next term is defined using the previous terms. By far the most famous of these is the Fibonacci sequence:

$$F_0 = 0, F_1 = F_2 = 1, \qquad F_{n+1} = F_n + F_{n-1} \tag{1}$$

The first several terms of this sequence are below:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34, \dots$$

For such sequences there is a method of finding a general formula for $n$th term, outlined in problems below.

**1.** Let us call a sequence $a_n$ a *generalized Fibonacci sequence* (GFC) if it satisfies the same recurrence relation ($a_{n+1} = a_n + a_{n-1}$), but might have different first two terms.

(a) Show that a geometric progression $a_n = \lambda^n$, $\lambda \neq 0$, is a GFC if and only if $\lambda$ satisfies the equation

$$\lambda^2 = \lambda + 1 \tag{2}$$

Find the roots of this equation.

(b) Let $\lambda_1, \lambda_2$ be the two roots of equation (2). Show that then any sequence of the form

$$a_n = c_1\lambda_1^n + c_2\lambda_2^n \tag{3}$$

(where $c_1, c_2$ are some constants that do not depend on $n$) is a GFC.

(c) Find constants $c_1, c_2$ so that the sequence $a_n$ defined by (3) satisfies $a_0 = 0$, $a_1 = 1$.

(d) Write a general formula for $F_n$.

**2.** Use a calculator to estimate how large $F_{1000}$ is.

**3.** Prove that the Fibonacci numbers satisfy the following identity: $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$

Can you guess a formula for the sum $F_1^2 + F_2^2 + \cdots + F_n^2$?

**4.** Pell numbers are defined by the relations

$$P_0 = 0, P_1 = 1, \qquad P_{n+1} = 2P_n + P_{n-1} \tag{4}$$

Try to modify the method of problem 1 to get a formula for Pell's numbers.

**5.** Show that for large $n$, the ratio $(P_{n-1} + P_n)/P_n$ is close to $\sqrt{2}$. Write the approximation one gets in this way for $n = 8$ and check how close it is to the actual value. (This series of approximations to $\sqrt{2}$ was known already in 4th century BC).

**6.** (This problem suggested by Alexander Kirillov Sr, is a natural continuation of the problem from previous assignment).

In 1950s, there were seven high rise buildings in the city of Moscow (one of them was Moscow State University). You could see them from any point in the city. Of course, the order in which these buildings appear on the skyline would depend on the observation point. [here the order means a cyclic order, that is, a way of arranging objects on a circle]. Is it true that all possible cyclic orders of these buildings could be observed from some point in the in the city?

You can assume that no three of these buildings were on the same line.