Homework 13
Fibonacci Numbers and the Golden Ratio

Math 7a

January 9, 2018

The Fibonacci sequence is a sequence of integers in which the first and second terms are both equal to 1 and each subsequent term is the sum of the two preceding it. The first few terms are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$. The sequence is described by: $F_0 = F_1 = 1$, $F_2 = F_0 + F_1$, and so on: $F_n = F_{n-2} + F_{n-1}$.

1. Show that for any integer n, the following equality is true:

$$F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$$

   [*Hint* : $F_{n+2} = F_{n+1} + F_n$.]

2. Divisibility of Fibonacci numbers
   (a) Which Fibonacci numbers are even? Give an explanation. [*Hint* : track the remainders.]
   (b) Which Fibonacci numbers are divisible by 3? Explain why.

3. Let us define some constants:

$$\Phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

$$\overline{\Phi} = \frac{1-\sqrt{5}}{2} \approx -0.618$$

   (a) Show that $\Phi^2 = \Phi + 1$ and $\overline{\Phi}^2 = \overline{\Phi} + 1$ are true.
   (b) Show that the geometric progression $a_1 = 1$, $a_2 = \Phi$, $a_3 = \Phi^2$, $\ldots$ satisfies the exact same rule as the Fibonacci sequence:

$$a_{n+2} = a_{n+1} + a_n \text{ or } \Phi^{n+2} = \Phi^{n+1} + \Phi^n$$

   [*Hint* : use previous part.]

4. Find $\gcd(F_n, F_{n+1})$.

5. Imagine you are tossing a single coin, but many times getting sequences of outcomes. For example we may get $HTTH$, or $TTHT$ if we toss the coin $n = 4$ times where H stands for heads and T stands for tails. We discard all sequences where 2 consecutive heads were present as illegal. For example $HTHH$, $HHHT$ and $THHT$ are not allowed if we flipped 4 times. But $HTHT$ and $TTTT$ are legitimate outcomes.
   (a) How many different sequences of allowed outcomes can we get if we to do 5 coin-flip experiments?
   (b) Calculate how many different sequences of allowed outcomes (no 2 consecutive heads) are possible if we flip the coin 15 times? This should not involve listing all possible sequences of 15 coin-flip outcomes.
   [*Hint* : consider the last (15th) coin-flip. We know something about each case of its outcome.]
   (c) Write down the formula for number of legitimate outcomes possible for n coin-flips.