LOOKING FOR INTEGER AND RATIONAL ROOTS

A definition defines or explains what a term means. Theorems/Properties/"Facts" must be proven to be true based on postulates and/or already-proven theorems.

Definition 1
A rational expression (function) : $f(x) = \frac{p(x)}{q(x)}$*, where* $p(x)$ *and* $q(x)$ *are polynomials (functions) and* $q(x) \neq 0$*.*

Definition 2
The ***domain of a rational expression*** *is the set of values that when substituted into the expression produces a (real) number. Therefore, the domain of a rational expression must exclude the values that make the denominator zero (as we do not divide by zero).*

Example 1

- Find the domain of $R(x) = \frac{x}{x^4-4}$

  Sol: $x^4-4 = x^4-2^2 = (x^2-2)(x^2+2) = (x-\sqrt{2})(x+\sqrt{2})(x^2+2)$ so the domain of $R(x)$ is $x \in \mathbb{R}|x \neq \sqrt{2}$ and $x \neq -\sqrt{2} = \mathbb{R} - \{-\sqrt{2}, \sqrt{2}\}$

- find the domain of $R(t) = \frac{t^3+8}{t^2+6t+8}$, simplify it and solve $R(t) = 0$

  Sol : $t^2 + 6t + 8 \neq 0$ Factor $t^2 + 6t + 8$ by finding two numbers whose product is 8 and whose sum is 6. The factors of 8 that sum to 6 are 4 and 2. So, $t^2 + 6t + 8 = (t + 4)(t + 2)$

  So, $t^2 + 6t + 8 \neq 0$ becomes $(t + 4)(t + 2) \neq 0$ iff $t \neq -4$ and $t \neq -2$

  The domain of $R(t)$ is $t \in \mathbb{R}|t \neq -4$ and $x \neq -2 = \mathbb{R} - \{-4, -2\}$

  Simplification:

  Using the factoring of the denominator we have $R(t) = (t^3 + 8)/((t + 4)(t + 2))$

  Hint: Express t^3+8 as a sum of cubes: $t^3+8 = t^3+2^3$: Factor the sum of two cubes using $a^3+b^3 = (a+b)(a^2-ab+b^2)$
  Thus, $t^3 + 2^3 = (t + 2)(t^2 - t2 + 2^2)$ The rational expression becomes:

  $$\frac{t^3 + 8}{t^2 + 6t + 8} = \frac{(t + 2)(t^2 - 2t + 4)}{(t + 4)(t + 2)}$$

  We can cancel the common terms in the numerator and denominator because they are non-zero.

Theorem 1
(Integer Root Test) Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ a polynomial with leading coefficient 1 and integer coefficients. If k is an integer root (i.e. $P(k) = 0$), then k is a factor of a_0.

Proof: $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = (x - x_1)(x - x_2)...(x - x_{n-1})(x - k)$ Two polynomials are equal if they their coefficients are identical. Their free terms (i.e. free of x) are $a_0 = x_1 \times x_2 \times \cdot \times k$. Thus, k is a facor of a_0

Theorem 2
(Rational Roots Test) Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ a polynomial where $a_n \neq 0$ and the a_i are integers. If p and q are relatively prime integers and $x = \frac{p}{q}$ is a root (i.e. $f(p/q) = 0$), then q is a factor a_n and p is a factor of a_0.

Proof (Optional Reading for the interested. Not done in class):

For simplicity let us prove for n=4. The same proof holds for general n. By assumption, $0 = f(p/q) = a_4(p/q)^4 + a_3(p/q)^3 + a_2(p/q)^2 + a_1(p/q) + a_0$. Multiplying the equation by q^4 we find that $0 = a_4p^4 + a_3qp^3 + a_2q^2p^2 + a_1q^3p + a_0q_4$. We rewrite as $a_0q_4 = -a_4p^4 - a_3qp^3 - a_2q^2p^2 - a_1q^3p = p(-a_4p^3 - a_3qp^2 - a_2q^2p - a_1q^3)$ Thus, p is a factor of a_0q_4. But p and q are relatively prime, so p divides a_0. Similarly, we can rewrite $0 = a_4p^4 + a_3qp^3 + a_2q^2p^2 + a_1q^3p + a_0q_4$. to isolate $a_4p^4 = -(a_3qp^3 + a_2q^2p^2 + a_1q^3p + a_0q_4) = -q(a_3p^3 + a_2qp^2 + a_1q^2p + a_0q_3)$. Thus, q divides a_4p^4. But p and q are relatively prime, so q divides a_4.

Example: Look for the rational roots of $p(x) = 2x^3 + 3x - 5$.

The constant term is 5, so its factors are $\pm 1, \pm 5$. The leading coefficient is 2, so its factors are $\pm 1, \pm 2$. $gcd(2, 5) = 1$ so they are relatively prime numbers.

Using the Rational Roots Test Th. the possible solutions are in the set $\{\pm 1, \pm\frac{1}{2}, \pm 5, \pm\frac{5}{2}\}$. Only $P(1) = 0$.

Recall

The rational function defined by $y = f(x) = \frac{1}{x}$ has a restriction on its domain that $x \neq 0$ and its graph has a vertical asymptote at $x = 0$

Homework

1. Let $p(x) = 2x^4 - 3x^2 - x + 2$. What are its possible rational roots? Find the remainder when $p(x)$ is divided by $(x - 3)$, $(x + 1)$, $(x - 1/2)$, and $(x + 1)$.
2. Which of the following are factors of $p(x) = x^3 - 6x^2 + 11x - 6?(x - 2)$,$(x + 1)$, or $(x - 1)$. Guess as many factors as possible and afterwards divide to write $p(x)$ as a product of factors.
3. Factorize in real numbers the quadratic $x^2 + 7x - 1$ and the quartic $x^4 + 7x^3 - 2x^2 - 7x + 1$.
4. Explain why we cannot factorize in real numbers the quadratic $x^2 + 2x + 4$. Factorize the polynomial $p(x) = x^4 - 2x^3 - 8x + 16$.
5. Solve algebraically and graphically the following polynomial systems:
   (a) $x^2 + y^2 = 1, xy = 1$
   (b) $x^2 + y^2 = 1, xy = 1/2$
   (c) $x^2 + y^2 = 1, xy = 1/4$
   (d)* What can you say in general about $x^2 + y^2 = 1, xy = k > 0$
6. Solve algebraically and graphically the following polynomial system $x^2 + y^2 = 1, x^2 - 4x + y^2 - 2y + 5 = 4$

7.* Write a polynomial expression to solve the problem of a Macedonian army commander from 360BCE:

"In a confrontation a Macedonian company was advancing through the battle field as two squared phalanges with the commander in front of them. In front of the enemy they regrouped themselves as a new squared phalange with the commander included in the battle formation. As Alexander the Great advances he looses troops. What is the minimal number of soldiers that a Macedonian company has to have in order to perform their advancing and battle formations? "

8.* Find the remainder of $x^{81} + x^{49} + x^{25} + x^9 + x$ by $x^3 - x$.

9.* For which $n \geq 3$ is it possible to inscribe a regular n-gon in an ellipse that is not a circle? (The equation of an eclipse has degree 2 and equals $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$)

10.* Let $P(x) = x^{2018} + a_{2017}x^{2017} + \cdots + a_1x + a_0$ be a polynomial with integer coefficients. Let four distinct integers k, l, m, n such that P(k) = P(m) = P(n) = P(l) = 5 then there is no integer k with P(k) = 8.