CHAPTER 7

QUADRATIC FUNCTIONS: MINIMUM/MAXIMUM POINTS, USE OF SYMMETRY

7.1 Minimum/Maximum, Recall: Completing the square

The "completing the square" method uses the formula

$$(x+y)^2 = x^2 + 2xy + y^2$$

and forces a "square" out of the terms in x^2 and x, and tries to rewrite the quadratic polynomial as $a(x-h)^2+k$
For instance, we can rewrite: $x^2+6x+2 = x^2+2\cdot 3x+9-7 = (x+3)^2-7$, and then, $x^2+6x+2=0$ if and only if $(x+3)^2=7$, which gives $x+3=\sqrt{7}$ or $x+3=-\sqrt{7}$. Thus the roots are $x_1=-3-\sqrt{7}$ and $x_2=-3+\sqrt{7}$.
So, for $a=1$, we can write

$$x^2+bx+c = x^2+2\cdot\frac{b}{2}\cdot x+c = x^2+2\cdot\frac{b}{2}\cdot x+\frac{b^2}{2^2}-\frac{b^2}{2^2}+c = \left(x+\frac{b}{2}\right)^2-\frac{D}{4},\ \text{where } D=b^2-4c.$$

For the general case ($a\neq 0$ is not 1), we can first divide everything by a, to equivalently solve

$$a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=0 \Longleftrightarrow a\left(\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a^2}\right)=0 \text{ and now } D=b^2-4ac$$

The expression

$$\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a^2},\ \text{where } D=b^2-4c.$$

shows that the minimum (or maximum in the case when a is negative) occurs when the first bracket is zero, that is, when $x=-\frac{b}{2a}$.
Recall that D is generally called the discriminant of the 2-nd degree equation because it is used to distinguish how many real number solutions the equation has.
In order to have solutions in real numbers, we need $D\geq 0$. Otherwise, if $D<0$, there are no real number solutions. So, when $D\geq 0$, we have a direct way of obtaining the solutions, called the discriminant formula

$$x+\frac{b}{2a}=\pm\frac{\sqrt{D}}{2a} \Longleftrightarrow x_{1,2}=\frac{-b\pm\sqrt{D}}{2a}$$

7.2 End behavior of a polynomial.

We have two cases x becomes larger, and larger in the positive direction, denoted by $x \to \infty$, and x becomes larger, and larger in the negative direction, denoted by $x \to -\infty$

Examples

- x^2,at $-\infty$: ,∞ : ,$-x^2$, at $-\infty$: ,∞ : , x^4,at $-\infty$: ,∞ : , $-x^4$, at $-\infty$: ,∞ :
- x^3, at $-\infty$: ,∞ : , $-x^3$, at $-\infty$: ,∞ : , x^5, at $-\infty$: ,∞ : , $-x^5$ at $-\infty$: ,∞ :

In general, the end behavior is determined by the term that contains the highest power of x, called the leading term. Why? Because when x is large, all the other terms are small compared to the leading term.

7.3 Turning point: What happens in the middle of the graph of a quadratic ?

In some locations the graph behavior changes. A turning point is a point at which the function values change from increasing to decreasing or decreasing to increasing. For the parabola it is also called the "vertex of the parabola".

Method 1

If we re-write the quadratic polynomial function as we did in the "completing the square" we have

$$y = f(x) = a(x-h)^2 + k \text{ , and its turning point is } (x, y) = (h, k)$$

or more precisely since

$$\left(x + \frac{b}{2a}\right)^2 = \frac{D}{4a^2} \text{ , and its turning point is } (x, y) = \left(-\frac{b}{2a}, \frac{D}{4a^2}\right)$$

Hint: Provided that you memorized the formula for the x_{vertex} of the vertex of the parabola its can be obtained just by plugging in the x_{vertex} into the given quadratic $y_{vertex} = f(x_{vertex}) = f\left(-\frac{b}{2a}\right)$.

Method 1 Complete the Square Examples for turning point/ parabola vertex:

- For x^2, if you memorized that $x_{vertex} = -\frac{b}{2a} = 0$ then $y_{vertex} = f(0) = (0)^2 = 0$
  If not it is square so, its turning point is $(x, y) = (0, 0)$.
- For x^2+3x+2, if you memorized that $x_{vertex} = -\frac{b}{2a} = -\frac{3}{2}$ then $y_{vertex} = f(-\frac{3}{2}) = (-\frac{3}{2})^2+3(-\frac{3}{2})+2 = -\frac{1}{4}$ If not complete the square to obtain: $y = x^2 + 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 2 = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + 2 = \left[x - \left(-\frac{3}{2}\right)\right]^2 - \frac{1}{4}$. So, the turning point is $(x, y) = \left(-\frac{3}{2}, -\frac{1}{4}\right)$.

Method 2 using symmetry

Due to the symmetry of the parabola, the turning point lies halfway between the x-intercepts. As a consequence, if there is only one x-intercept, then the x-intercept is exactly the turning point.

The x-intercepts are the roots of the quadratic polynomial. Let them be x_1, x_2.
What is the midpoint of the segment determined by the roots x_1, x_2?
According to the formula of the midpoint: $x_{vertex} = \frac{x_1+x_2}{2}$
Using Vieta's formulas: $x_{vertex} = \frac{x_1+x_2}{2} = -\frac{b}{2a}$

Method 2: Symmetry, Examples for turning point/ parabola vertex:

x^2, x^2+4x+4, x^2+3x+2

- The x-intercept is found by letting $y = f(x) = 0$. So, $0 = x^2 \implies x = 0$. There is only one x-intercept, so that gives the x-coordinate of the turning point. To find the y-coordinate, substitute it into the quadratic equation. $f(0) = 0^2 \implies y = 0$. Thus, the turning point is (x, y) = (0, 0).
- $x^2+4x+4 = (x+2)^2$ There is only one x-intercept, so that gives the x-coordinate of the turning point.
- $x^2+3x+2 = (x+1)(x+2)$. So, there are two roots, thus two distinct x-intercepts: $x = -1$ or $x = -2$. To find the x-coordinate of the turning point, average the x-intercepts. So, $x_{tp} = \frac{(-1)+(-2)}{2} = -\frac{3}{2}$. To find its y-coordinate, substitute this value into the equation: $y_{tp} = \left(-\frac{3}{2}\right)^2 + 3\left(-\frac{3}{2}\right) + 2 = \frac{9}{4} - \frac{9}{2} + 2 = -\frac{1}{4}$. Thus, the coordinates of the turning points are $(x, y) = \left(-\frac{3}{2}, -\frac{1}{4}\right)$.

7.4 Axis of symmetry

Symmetry can be useful in graphing an equation since it says that if we know one portion of the graph then we will also know the remaining (and symmetric) portion of the graph as well.

A function f which is symmetric with respect to the y-axis, is called even and satisfies f(x) = f(-x). Similarly, an odd function is rotationally symmetric about the origin and satisfies f(x) = -f(-x). For polynomials, there is an easy way to tell the difference: even polynomials only have even power terms; odd polynomials have odd power terms. A polynomial with a mix of terms, as is typical, is neither even nor odd. Their the centers/axis of symmetry are shifted. Where do we identify this shift for a quadratic? In its (h,k) expression (after completing to a square.

We can complete the square on the general quadratic polynomial function and thereby obtain a general formula for the axis of symmetry and hence the x-coordinate for the vertex.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{D}{4a^2} \text{ , and its vertex is } (x, y) = \left(-\frac{b}{2a}, \frac{D}{4a^2}\right)$$

The axis of symmetry of a quadratic is given by y=x-coordinate of the vertex of the parabola/its turning point.

7.5 Classwork-Worked out example

Given the polynomial function $y = f(x) = x^2 + x - 2$ determine the leading term, degree, and its end behavior. Determine its x-intercepts if any, the turning point/ parabola vertex and try to sketch its graph.

the leading term, degree, and its end behavior:

The leading term: x^2, degree:2 End behaviour at $-\infty : +\infty \quad ,\infty : +\infty$

Find the real x-intercepts(roots) if any:

$$y = f(x) = x^2 + x - 2$$

Find the roots of the polynomial function of second degree (that is, the zeros of the quadratic equation):

$$f(x) = (x+2)(x-1)$$

The roots are x-intercepts: $(-2, 0)$, $(1, 0)$.

Axis of symmetry and turning point:

Halfway between the x-intercepts:

$$x_t = -\frac{b}{2a} = -\frac{1}{2} = -0.5$$

$$\Longrightarrow y_t = (-0.5)^2 - 0.5 - 2 = -0.25 - 2 = -2.25$$

End behavior:

$$x \to \infty \Longrightarrow f(x) \to \infty, f(x) > 0$$

$$x \to -\infty \Longrightarrow f(x) \to \infty, f(x) > 0$$

x	$-\infty$		-2		-0.5		$+1$		$+\infty$
$(x+2)$									
$(x-1)$									
$f(x) = (x+2)(x-1)$									

We take test points in these intervals to determine the signs.
For example, test point for

$$\begin{array}{lcl} (-\infty, -2) & : & x = -3 = -0.5 - 2.5 \\ (-2, -0.5) & : & x = -1 = -0.5 - 0.5 \\ (-0.5, 1) & : & x = 0 = -0.5 + 0.5 \text{the symmetrical point such that the y-values are known} \\ (1, \infty) & : & x = -0.5 + 2.5 = 2 \text{the symmetrical point such that the y-values are known} \end{array} \Longrightarrow$$

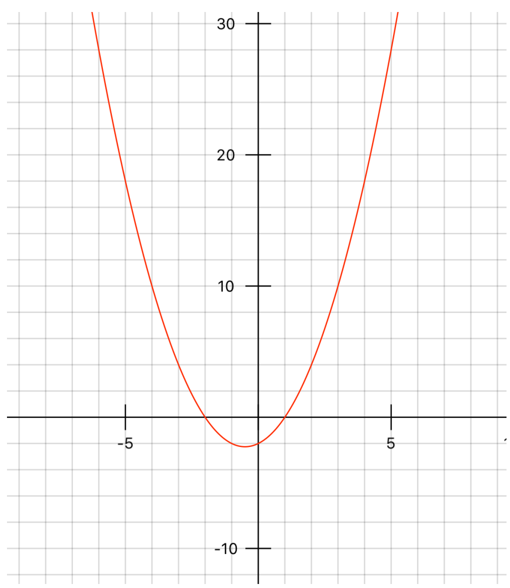

Figure 7.1: Graph of $f(x) = (x+2)(x-1)$

Example of using the zeroes to graph a quadratic polynomial function

x	$-\infty$	-3	-2	-1	-0.5	0	$+1$	$+2$	$+\infty$
$(x+2)$	$-\infty$	-1	0	$+1$	$+1.5$	$+2$	$+3$	$+4$	$+\infty$
$(x-1)$	$-\infty$	-4	-3	-2	-1.5	-1	0	$+1$	$+\infty$
$f(x)=(x+2)(x-1)$	$+\infty$	$+4$	0	-2	-2.25	-2	0	$+4$	$+\infty$

7.6 Use the graph for applications

A very inexperienced tennis player keeps throwing balls over the fence. The height in the air of his last thrown ball after x seconds is determined by the equation $y = h(x) = -16x^2 + 80x + 96$.

- Determine the highest point that the ball will reach.
- Determine when the ball will be at 160 feet high
- Determine when the ball will land.

Hints:

Determine the highest point that the ball will reach.

What type of a parabola do we have? (regular, upside down)
Does it have a minimum or maximum point?
For which x do we reach the minimum or maximum point?

Determine when the ball will be at 160 feet high

The height is 160, so $y = h(x) = 160$, so solve $h(x) = -16x^2 + 80x + 96 = 160$
How many roots can you expect? Why?

Determine when the ball will land.

Landing corresponds to the height $y_0 = 0$?
Solve the corresponding equation: $y_0 = h(x) = -16x^2 + 80x + 96$
It lands once. If it restarts jumping afterwards we consider it a different motion. How many roots can you expect? (Very often you will
What to do?

7.7 Homework

1. Find the vertices of the following parabolas
   - x^2, roots: , Vieta's sum of the roots: midpoint=vertex of the parabola:
   - $x^2 + 4x + 4$,roots: , Vieta's sum of the roots: midpoint=vertex of the parabola:
   - $x^2 + 3x + 2$, roots: , Vieta's sum of the roots: midpoint=vertex of the parabola:
2. Find the coordinates of the vertex of the parabola
   - $-x^2$, $x_{vertex} =$ $y_{vertex} = f(x_{vertex}) =$,
   - x^2+2x, $x_{vertex} =$ $y_{vertex} = f(x_{vertex}) =$, $-x^2+2x$, $x_{vertex} =$ $y_{vertex} = f(x_{vertex}) =$,

- x^2+2, $x_{vertex}=$ $y_{vertex}=f(x_{vertex})=$, $-x^2+2$, $x_{vertex}=$ $y_{vertex}=f(x_{vertex})=$,
- x^2+2x+4, $x_{vertex}=$ $y_{vertex}=f(x_{vertex})=$, x^2+3x+2 $x_{vertex}=$ $y_{vertex}=f(x_{vertex})=$

3. The cost of x days of work from the construction company A is $C=\frac{x^2}{5}$. The cost for x days of work of construction company B is $C=\frac{2x}{5}+3$. Determine the number of days that the cost to the construction company A is equal to the cost of using the construction company B.

4. A stunt man drops from the top of a 50 foot elevator shaft. His height above ground as a function of time is determined by the equation $y=h(t)=-16t^2+50$. At the moment the stunt man begins to drop, an elevator goes up so that its height above ground is determined by the equation $h(t)=34t$. After how many seconds will the stunt man land on top of the elevator ?

5. A tennis player throws balls. The height in the air of his last thrown ball after x seconds is determined by the equation $y=h(x)=-x^2+x+2$.
   - Determine the highest point that the ball will reach.
   - Determine when the ball will be at 2 feet high
   - Determine when the ball will land.

6. Graph the functions finding the behavior at ∞ and $-\infty$, the turning points and the axis of symmetry:
   (a) $x^2+x+6=0$
   (b) $(x-2)(x+18)=0$
   (c) $x^2+x=0$
   (d) $x^2-5x+6=0$

7. Solve the inequalities using a table similar to one used for graphing quadratic functions
   (a) $\frac{2x+1}{x-5}\leq 0$
   (b) $|x^2-x|>1$