CHAPTER 8

QUADRATICS: PRACTICE ON COMPLETING THE SQUARE

If the equation does not have integer solution it is difficult to factorize the quadratic using the sum and product of the roots (i.e. Vieta relations) and we use the "completing the square" method.

We know the roots of $z^2 = k$, for $k > 0$, i.e. $z_{1,2} = \pm\sqrt{k}$

The "completing the square" method tries to write a quadratic $x^2+bx+c=0$ as $(x-h)^2=k$, with $k>0$. This form is called the vertex form as $(h, -k)$ represents the minimum point of the parabola x^2+bx+c.

If we manage to write our quadratic in the vertex form the solutions became $x_{1,2} - h = \pm\sqrt{k}$

We can transform a quadratic x^2+bx into its "vertex form", i.e. "a perfect square - another constant term" by comparing x^2+bx with the formula of a perfect square:

$$(x+v)^2 = x^2 + 2vx + v^2$$

and "forcing" a "square" out of the terms in x^2 and bx

Example 1:

Rewrite x^2+6x in its "vertex form", i.e. "a perfect square - another constant term"

The coefficient of x is 6. If we compare $(x+v)^2 = x^2+2vx+v^2$ to x^2+6x we get $6x = 2vx$.

Dividing 6 by 2 we get $v = 3$. Thus,

$$x^2 + 6x = (x^2 + 6x + 9) - 9 = (x+3)^2 - 9$$

In a similar way we can also transform a quadratic x^2+bx+c into its "vertex form", i.e. "a perfect square - another constant term" by comparing once more x^2+bx with the formula of a perfect square:

$$(x+v)^2 = x^2 + 2vx + v^2$$

and "forcing" a "square" out of the terms in x^2 and bx

$$x^2+bx+c = x^2+2\cdot\frac{b}{2}\cdot x+c = x^2+2\cdot\frac{b}{2}\cdot x+\frac{b^2}{2^2}-\frac{b^2}{2^2}+c = \left(x+\frac{b}{2}\right)^2-\frac{D}{4}, \text{ where } D = b^2-4c.$$

Example 2:

Rewrite $x^2+6x-11$ in its "vertex form", i.e. "a perfect square - another constant term", and solve $x^2+6x-11=0$.

The coefficient of x is 6. If we compare $(x+v)^2 = x^2+2vx+v^2$ to x^2+6x we get $6x = 2vx$.

Dividing 6 by 2 we get $v = 3$. Thus,

$$(x^2+6x) - 11 = (x^2+6x+9) - 9 - 11 = (x+3)^2 - 9 - 11 = (x+3)^2 - 20$$

Now, we solve $x^2+6x-11=(x+3)^2-20=0$, so $(x+3)^2=20$. We denote $x+3$ by z and we have $z^2=20$, i.e. $z_{1,2}=\pm\sqrt{20}=\pm 2\sqrt{5}$. So, $x_{1,2}+3=\pm 2\sqrt{5}$. Finally, $x_{1,2}=-3\pm 2\sqrt{5}$
For the general case ($a\neq 0$ is not 1), we can first divide everything by a, to equivalently solve

$$a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=0 \Longleftrightarrow a\left(\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a^2}\right)=0 \text{ and now } D=b^2-4ac$$

Example 3:

Rewrite $2x^2+20x+1$ in its "vertex form", i.e. "a perfect square - another constant term" and solve $2x^2+20x+1=0$.

Rewrite the terms in x^2 and x to get x^2 with a coefficient 1.

$2(x^2+10x)+1$

The coefficient of x inside the parenthesis is 10. Divide 10 by 2.

$$2(x^2+10x)-11=2((x+5)^2-25)+1=2(x+5)^2-50+1=2(x+5)^2-49$$

Now, we solve $2x^2+20x+1=2(x+5)^2-49=0$, so $2(x+5)^2=49$. We denote $x+5$ by z and we have $2z^2=49$, i.e. $z_{1,2}=\pm\sqrt{\frac{49}{2}}=\pm\frac{49}{\sqrt{2}}$. So, $x_{1,2}+5=\pm\frac{49}{\sqrt{2}}$. Finally, $x_{1,2}=-5\pm\frac{49}{\sqrt{2}}$

Homework

1. Find x, if (2x+3)(x -9) = 0.
2. Write the following quadratics in their vertex form $a(x-h)^2+k$
   - (a) x^2+8x
   - (b) $x^2-x=0$
   - (c) $x^2+7x=0$
   - (d) $x^2+3x=0$
   - (e) $2x^2+8x=0$
   - (f) $9x^2+12x=0$
3. Write the following quadratics in the vertex form $(x-h)^2+k$
   - (a) x^2+8x+7
   - (b) $x^2-x+1=0$
   - (c) $x^2+7x-6=0$
   - (d) x^2+3x
   - (e) $2x^2+8x$
   - (f) $9x^2+12x$
4. The equation $x^2+7x-3=0$ is to be solved using the quadratic formula . What are the values of a, b and c?What is the value of the discriminant $D=b^2-4ac$?
5. Graph the functions finding the behavior at ∞ and $-\infty$, the turning points and the axis of symmetry:
   - (a) $x^2-4x+5=0$
   - (b) $x^2-x=0$
6. The cost of x days of work from the construction company A is $C=x^2+5$. The cost for x days of work of construction company B is $C=5x+11$. Is there a number of days for which it does not matter which company I choose? If yes what is that number of days?
7. A tennis machine throws balls. The height in the air of his last thrown ball after x seconds is determined by the equation $y=h(x)=-x^2+8x+1$.
   - Determine the highest point that the ball will reach.
   - Determine when the ball will be at 10 feet high.
   - Determine when the ball will land.