# MATH 7
# HOMEWORK 4: VECTORS - APPLICATIONS
OCT 5, 2014

## Relativity of velocity

The following rule is sometimes called Galileo relativity principle:

If point $A$ is moving with velocity $\vec{v}_{rel}$ *relative to* $B$ (i.e., observer at $B$ sees $A$ moving with this velocity) and $B$ itself is moving with velocity $\vec{v}_B$, then the velocity of $A$ is equal to $\vec{v}_A = \vec{v}_B + \vec{v}_{rel}$.

It can also be used to find the relative velocity: if $A$ is moving with velocity $\vec{v}_A$, and $B$ with velocity $\vec{v}_B$, then the velocity of $A$ as seen by observer at $B$ is equal to $\vec{v}_{rel} = \vec{v}_A - \vec{v}_B$.

## Application: center of gravity

For a collection of points $A_1, \dots, A_n$ and positive numbers $m_1, \dots, m_n$ (masses placed at these points), we define the center of gravity of this collection of points to be a point $M$ such that

$$\overrightarrow{OM} = \frac{m_1\,\overrightarrow{OA_1} + \cdots + m_n\,\overrightarrow{OA_n}}{m_1 + m_2 + \cdots + m_n}$$

It can be shown that this definition does not depend on the choice of point $O$: if we choose another point $O'$ and define $M'$ so that $\overrightarrow{O'M'} = \frac{m_1\overrightarrow{O'A_1} + \cdots + m_n\overrightarrow{O'A_n}}{m_1+m_2+\cdots+m_n}$ then in fact $M = M'$.

Examples:

- Center of gravity of two points $A, B$ with equal mass at them is the midpoint of the interval $AB$.
- Center of gravity of the four vertices of the parallelogram is the intersection point of its diagonals.

## Problems

**1.** A ship is sailing east with speed 10 mph. A person is walking on the deck of the ship going north with speed of 3 mph. What is the velocity of this person relative to the shore? Write it as a vector in the coordinate system where $y$ axis is pointing north, and $x$ axis is pointing east.

**2.** Consider the system of 5 points moving in the plane. At a given moment, the coordinates and velocities of these points are as shown below:

$A(2, 0)$; $\vec{v}_A = (4, 0)$
$B(3, 0)$; $\vec{v}_B = (6, 0)$
$C(0, -1)$; $\vec{v}_C = (0, -2)$
$D(-1, 1)$; $\vec{v}_D = (-2, 2)$
$E(1, 3)$; $\vec{v}_E = (2, 6)$

(as you can see, each point is moving in a straight line away from the origin, and the velocity is proportional to the distance from the origin).

What would the observer at point $A$ see? what would be velocities of points $B$, $C$, $D$, $E$ measured from $A$? Draw these velocity vectors on the plane.

**3.** (a) Let masses $m_1 = 3$, $m_2 = 1$ be placed at points $A_1 = (3, 6)$, $A_2 = (11, 2)$. Find the center of gravity of these two masses. Does it lie on the segment $A_1A_2$? in what proportion does it divide it?

(b) Consider the center of gravity $M$ of a system of two masses $m_1, m_2$ at points $A_1, A_2$. Prove that then $\overrightarrow{A_1M} = \frac{m_2}{m_1+m_2}\overrightarrow{A_1A_2}$. Can you write a similar formula for $\overrightarrow{MA_2}$?

(c) Prove that the center of gravity $M$ of a system of two masses $m_1, m_2$ at points $A_1, A_2$ lies on the segment $A_1A_2$ and divides it in proportion $m_2 : m_1$.

**4.** (a) Let $M$ the center of gravity of three points $A, B, C$ with unit mass at each of them. Prove that then

$$\overrightarrow{OM} = \tfrac{1}{3}\overrightarrow{OA} + \tfrac{2}{3}\overrightarrow{OA_1}$$

where $A_1$ is the midpoint of $BC$.

(b) Prove that all three medians of a triangle intersect at a single point $M$ which divides each of them in proportion $2 : 1$

***5.** Consider a triangle $\triangle ABC$ and let

(a) $A_1$ be the point on side $BC$ which divides it in proportion $2 : 3$,

(b) $B_1$ be the point on side $CA$ which divides it in proportion $3 : 4$,

(c) $C_1$ be the point on side $AB$ which divides it in proportion $2 : 1$

Prove that the lines $AA_1$, $BB_1$, $CC_1$ all intersect at a single point. [Hint: this point would be the center of gravity of appropriately chosen 3 masses at points $A, B, C$.]

**6.** Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$ be four vertices of a quadrilateral.

(a) Write a formula for vectors $\overrightarrow{A_1B_1}$, $\overrightarrow{D_1C_1}$, where $A_1, B_1, C_1, D_1$ are midpoints of sides $AB$, $BC$, $CD$, $DA$ respectively.

(b) Prove that $\overrightarrow{A_1B_1} = \overrightarrow{D_1C_1}$

(c) Prove that in any quadrilateral, midpoints of 4 sides form a parallelogram.