Transformations of the plane

A transformation T is an operation which sends every point P of the plane to a new point, $T(P)$.

A composition of two transformations T_1, T_2 is the operation obtained by doing first T_2 and then T_1 (note the order!):

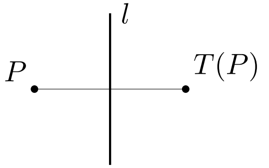
$$T_1 \circ T_2(P) = T_1(T_2(P))$$

The identity transformation, usually denoted by I, is the one that "does nothing", i.e. sends every point to itself: $I(P) = P$

An inverse of the transformation T is the transformation T^{-1} such that $T^{-1} \circ T = T \circ T^{-1} = I$: if $T(P) = Q$, then $T^{-1}(Q) = P$.

Here are some examples of transformations:

Reflection: For any line l, the reflection R_l is defined by the condition that $T(P)$ lies on the perpendicular from P to l, on the other side of l than P, at the same distance from l.



Rotation: For any point O and real number φ, we denote by $R_{O,\varphi}$ the counterclockwise rotation around O by the angle φ (if φ is negative, $R_{O,\varphi}$ is actually a clockwise rotation). An important special case is when $\varphi = 180°$; in this case, this transformation is sometimes called "symmetry around point O".

Note that $R_{O,\varphi}$ only depends on φ modulo $360°$: $R_{O,\varphi} = R_{O,\varphi+360°}$.

Translation: Given a vector $\vec{v}$, we define the translation $T_{\vec{v}}$ to be the operation that adds to each point the vector $\vec{v}$: a point P is sent to a point P' so that $\overrightarrow{PP'} = \vec{v}$.

In other words, every point P is moved in the same direction and by the same distance, given by vector $\vec{v}$.

In fact, these transformations have some special property:

A transformation is an isometry if it preserves distances: for any points P, Q, we have $T(P)T(Q) = PQ$

Theorem.

1. *Reflections, rotations, and translations are isometries.*
2. *Any isometry sends lines to lines: if l is a line and T an isometry, then $T(l)$ is again a line.*
3. *Composition of isometries is again an isometry*

We will not prove it here.

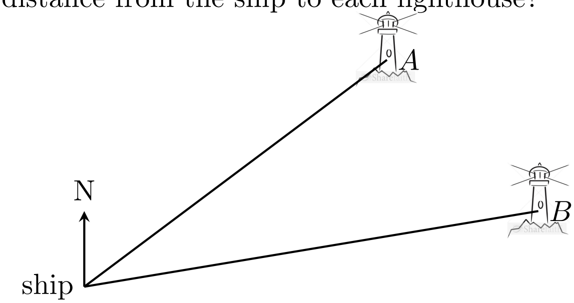
Homework

In all of these problems, you can use a calculator to compute sines and cosines.

1. A ship at sea tries to determine its position by measuring azimuths of two lighthouses, A and B. ["azimuth", or bearing, is the angle between the true north direction and the ray connecting the ship to the object – in our case, the lighthouse; it is measured clockwise.]

They found the azimuth of lighthouse A to be 40° and azimuth of lighthouse B to be 70°.

If it is also known that lighthouse A is exactly 60 miles northwest of lighthouse B, can you determine the distance from the ship to each lighthouse?



[Hint: use law of sines!]

2. Let $T = R_l$ be the reflection around the x-axis. For each of the following points P, find the coordinates of the corresponding point $T(P)$.

(a) $P_1 = (1, 1)$ (b) $P_2 = (2, 3)$ (c) $P_3 = (-3, 0)$ (d) $P_4 = (5, -1)$

Can you write a general formula: if $P = (x, y)$, then $T(P) =??$

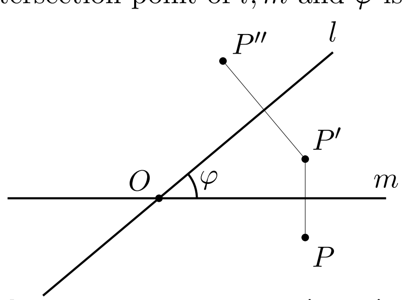
3. Answer the same questions for reflection around the line $x = y$.

4. Answer the same questions for reflection around the line $x = 1$.

5. Answer the same questions for 90° rotation around the origin.

6. Answer the same questions for the translation $T_{\vec{v}}$, where $\vec{v} = (3, 4)$.

***7.** Let l, m be two intersecting lines. Prove that then $R_l \circ R_m = R_{O,2\varphi}$, where O is the intersection point of l, m and φ is the angle between them.



[Hint: triangles $\triangle POP', \triangle P'OP''$ are isosceles.]

***8.** Can you guess what is a composition of a translation and a reflection? Is it a translation? reflection? neither?