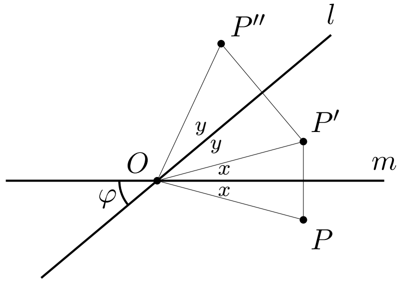
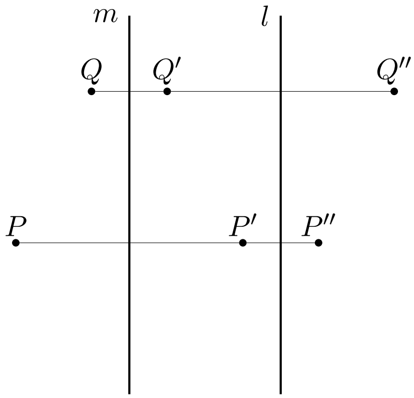
Compositions of transformations

Theorem 1. *Let l, m be two intersecting lines. Then the composition of two reflections R_l, R_m is a rotation: $R_l \circ R_m = R_{O,2\varphi}$, where O is the intersection point of l, m and φ is the angle between them.*



Proof. Since reflections preserve distances, $OP = OP' = OP''$. Since the triangle OPP' is isosceles, the two angles labeled by x in the figure are equal; similarly the two angles labeled by y are equal. Therefore, $\angle POP'' = 2x + 2y = 2(x + y) = 2\varphi$. □

Theorem 2. *Let l, m be two parallel line which are distance d apart. Then $R_l \circ R_m$ is a translation by distance $2d$.*



Proof of this theorem is left to you as a homework exercise.

Theorem 3. *Composition of a reflection R_l and translation $T_{\vec{v}}$ in a direction perpendicular to l is again a reflection: $T_{\vec{v}} T_l = T_m$, where m is a line parallel to l.*

Proof. Let m be a line parallel to l and such the distance between them is half of length of $\vec{v}$. Then, by the previous theorem, one can write $T_{\vec{v}} = T_m T_l$. Thus, $T_{\vec{v}} T_l = T_m T_l T_l = T_m$. □

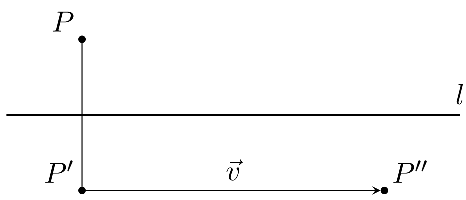
Theorem 4. *Let R_{O_1,φ_1}, R_{O_2,φ_2} be two rotations with different center O_1, O_2. Then the composition $R_{O_1,\varphi_1} R_{O_2,\varphi_2}$ is either a rotation or a translation.*

Proof. See homework problem 6 below. □

Slide reflections and classification

It turns out that there is one more type of isometry not discussed last time.

Slide reflection: Let $\vec{v}$ be a vector parallel to line l. Then the slide reflection is defined as composition $T_{\vec{v}} R_l$.



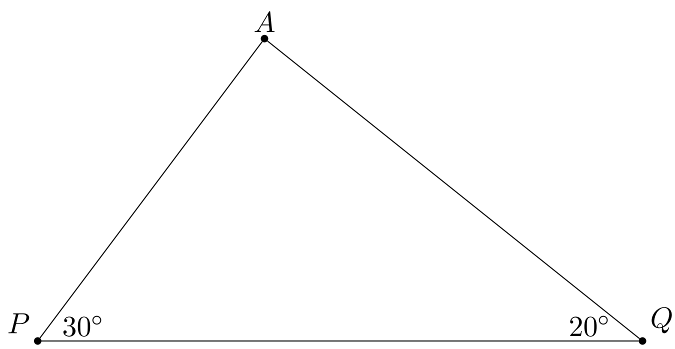
Theorem. *Any isometry of the plane is one of the four types listed before: rotation, reflection, translation, slide translation.*

In particular, it means that composing two transformations from this list, we again get a transformation from this list.

Homework

In all of these problems, you can use a calculator to compute sines and cosines.

1. Two radars P, Q are tracking an airplane A. The elevation angles (i.e., angle between the horizontal and the line to the airplane) are shown in the figure below. If the distance between the radars in 20 km, what is the altitude of the airplane? [Use law of sines!]



2. Prove Theorem 2. You only need to prove that PP'' is perpendicular to l and has length $2d$.

3. (a) Let l be the line $x = 1$. Find where the reflection R_l would send points $(2, 3)$; $(5, 0)$. Write the general formula: where it would send the point (x, y)? [Hint: it may help if you write $x = 1 + a$.]

(b) Same questions for reflection around the line m given by $x = 3$

(c) Compute the composition $R_l R_m$. Where would it send points $(2, 3)$; $(5, 0)$; (x, y)?

4. Prove that a composition of a reflection around line l and a rotation around the point O on this line is again a reflection. [Hint: use Theorem 1 to write rotation as a composition of two reflections.]

5. (a) Let A be the point $(2, 0)$ and let $R_{A,90^\circ}$ be the 90 degree (counterclockwise) rotation around A. Find where this rotation would send the the following points:
$P_1 = (4, 0)$
$P_2 = (2, 5)$
$P_3 = (3, 1)$
$P_4 = (2 + \sqrt{2}, \sqrt{3})$
Can you write the general formula: where this rotation would send point (x, y)? [Hint: it may be useful to write x in the form $x = 2 + a$.]

(b) Consider the composition $R_l R_{A,90^\circ}$, where l is the x-axis. FInd where it would send each of the points $P_1 - P_4$ above. Can you describe what kind of transformation this composition is: is it rotation, reflection, ...? around which point or line?

6. Prove Theorem 4. Hint: let l be the line through O_1, O_2; then one can write $R_{O_1,\varphi_1} = R_m R_l$, $R_{O_2,\varphi_2} = R_l R_n$ (note the order!) for suitably chosen lines m, n. Thus, $R_{O_1,\varphi_1} R_{O_2,\varphi_2} = R_m R_l R_l R_n$.

7. Consider the infinite pattern shown to the right. Can you describe all symmetries of it, i.e., all isometries which do not change this picture? [Of course, there are infinitely many of them; still, you may be able to give a description like "all translations by a vector from this (infinite) set."] Will the answer change if we draw a picture (say, a smiley face) in each parallelogram? if instead of parallelograms we had squares?