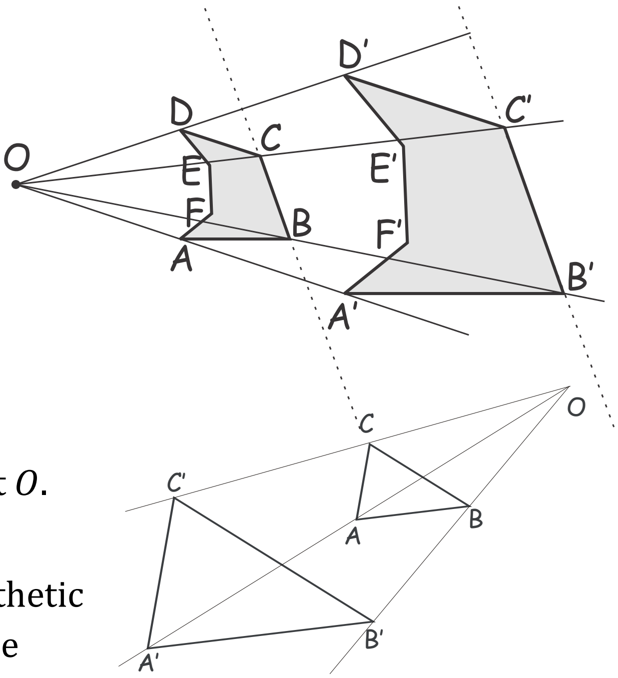
October 15, 2017

Geometry.

Similarity and homothety. Theorems and problems.

Definition. Two figures are homothetic with respect to a point O, if for each point A of one figure there is a corresponding point A' belonging to the other figure, such that A' lies on the line (OA) at a distance $|OA'| = k|OA|$ $(k > 0)$ from point O, and vice versa, for each point A' of the second figure there is a corresponding point A belonging to the first figure, such that A' lies on the line (OA) at a distance $|OA| = \frac{1}{k}|OA'|$ from point O. Here the positive number k is called the homothety (or similarity) coefficient. Homothetic figures are **similar**. The transformation of one figure (e.g. multilateral ABCDEF) into the figure A'B'C'D'E'F' is called homothety, or similarity transformation.

Homothety



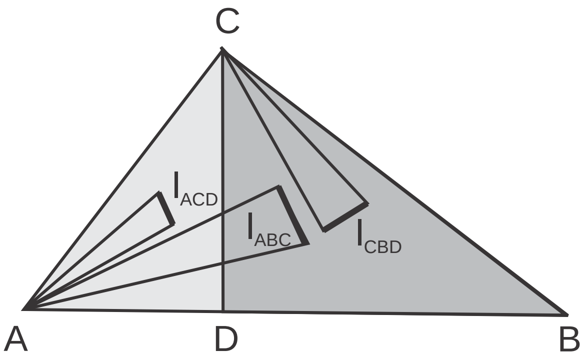
Thales Theorem Corollary 1. The corresponding segments (e.g. sides) of the homothetic figures are parallel.

Thales Theorem Corollary 2. The ratio of the corresponding elements (e.g. sides) of the homothetic figures equals k.

Exercise. What is the ratio of the areas of two similar (homothetic) figures?

Definition. Consider triangles, or polygons, such that angles of one of them are congruent to the respective angles of the other(s). Sides which are adjacent to the congruent angles are called *homologous*. In triangles, sides opposite to the congruent angles are also homologous.

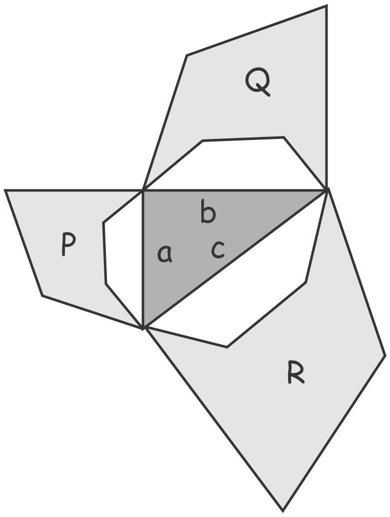
Generalized Pythagorean Theorem 2.



Theorem 2. For three homologous segments, l_{ABC}, l_{CBD} and l_{ACD} belonging to the similar right triangles ABC, CBD and ACD, where CD is the altitude of the triangle ABC drawn to its hypotenuse AB, the following holds,

$$l_{ACD}^2 + l_{CBD}^2 = l_{ABC}^2$$

Proof. If we square the similarity relation for the homologous segments, $\frac{l_{CBD}}{a} = \frac{l_{ACD}}{b} = \frac{l_{ABC}}{c}$, where $a = |BC|$, $b = |AC|$ and $c = |AB|$ are the legs and the hypotenuse of the triangle ABC, we obtain, $\frac{l_{CBD}^2}{a^2} = \frac{l_{ACD}^2}{b^2} = \frac{l_{ABC}^2}{c^2}$. Using the property of a proportion, we may then write, $\frac{l_{ACD}^2 + l_{CBD}^2}{a^2+b^2} = \frac{l_{ABC}^2}{c^2}$, wherefrom, by Pythagorean theorem for the right triangle ABC, $a^2 + b^2 = c^2$, we immediately obtain $l_{ACD}^2 + l_{CBD}^2 = l_{ABC}^2$.

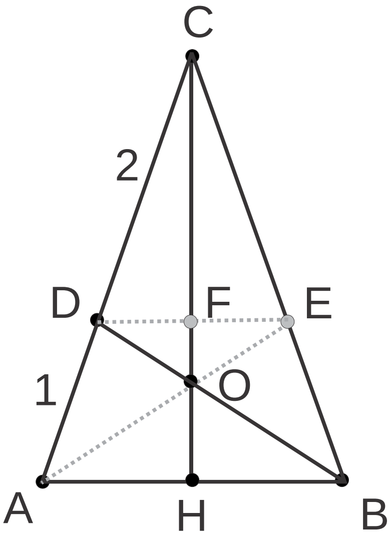


Theorem 1. If three similar polygons, P, Q and R with areas S_P, S_Q and S_R are constructed on legs a, b and hypotenuse c, respectively, of a right triangle, then,

$$S_P + S_Q = S_R$$

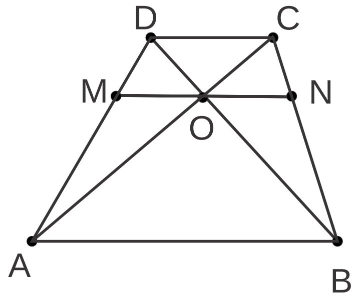
Proof. The areas of similar polygons on the sides of a right triangle satisfy $\frac{S_R}{S_P} = \frac{c^2}{a^2}$ and $\frac{S_R}{S_Q} = \frac{c^2}{b^2}$, or, $\frac{S_P}{a^2} = \frac{S_Q}{b^2} = \frac{S_R}{c^2}$. Using the property of a proportion, we may then write, $\frac{S_P+S_Q}{a^2+b^2} = \frac{S_R}{c^2}$, wherefrom, using the Pythagorean theorem for the right triangle ABC, $a^2 + b^2 = c^2$, we immediately obtain $S_P + S_Q = S_R$.

Selected problems on similar triangles.



Problem 1 (homework problem #4). In the isosceles triangle ABC point D divides the side AC into segments such that $|AD|:|CD| = 1:2$. If CH is the altitude of the triangle and point O is the intersection of CH and BD, find the ratio $|OH|$ to $|CH|$.

Solution. First, let us perform a supplementary construction by drawing the segment DE parallel to AB, $DE||AB$, where point E belongs to the side CB, and point F to DE and the altitude CH. Notice the similar triangles, $AOH \sim DOF$, which implies, $\frac{|OF|}{|OH|} = \frac{|DF|}{|AH|}$. By Thales theorem, $\frac{|AH|}{|DF|} = \frac{|AC|}{|AD|} = 1 + \frac{|CD|}{|AD|} = \frac{3}{2}$, and $\frac{|OF|}{|OH|} = \frac{|DF|}{|AH|} = \frac{2}{3}$, so that $\frac{|FH|}{|OH|} = \frac{|FO|+|OH|}{|OH|} = \frac{5}{3}$. $\frac{|CH|}{|OH|} = \frac{|CH|}{|FH|}\frac{|FH|}{|OH|} = 3 \cdot \frac{5}{3} = 5$, because $\frac{|CH|}{|FH|} = 1 + \frac{|CF|}{|FH|} = 1 + \frac{|CD|}{|DA|}$. Therefore, the sought ratio is, $\frac{|OH|}{|CH|} = \frac{1}{5}$.



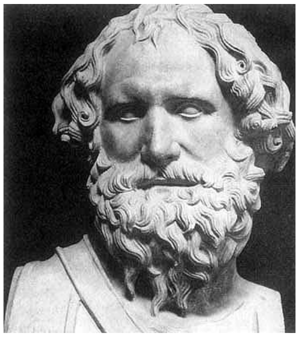
Problem 2 (homework problem #5). In a trapezoid $ABCD$ with the bases $|AB| = a$ and $|CD| = b$, segment MN parallel to the bases, $MN||AB$, connects the opposing sides, $M \in [AD]$ and $N \in [BC]$. MN also passes through the intersection point O of the diagonals, AC and BD, as shown in the Figure. Prove that $|MN| = \frac{2ab}{a+b}$.

Solution. By Thales theorem applied to vertical angles AOB and DOC and parallel lines AB and CD, $\frac{|AM|}{|MD|} = \frac{|BN|}{|NC|} = \frac{|AB|}{|DC|} = \frac{a}{b}$. Consequently, $\frac{|AD|}{|MD|} = \frac{|AM|+|MD|}{|MD|} = \frac{a}{b} + 1 = \frac{|BN|+|NC|}{|NC|} = \frac{|BC|}{|NC|}$. Now, applying the same Thales theorem to angles ADB and ACB and parallel lines MN and AB, we obtain, $\frac{|MO|}{|AB|} = \frac{|MD|}{|AD|} = \frac{1}{\frac{a}{b}+1}$ and $\frac{|ON|}{|AB|} = \frac{|NC|}{|BC|} = \frac{1}{\frac{a}{b}+1}$. Hence, $\frac{|MO|}{|AB|} + \frac{|ON|}{|AB|} = \frac{|MN|}{|AB|} = \frac{2}{\frac{a}{b}+1}$, and $|MN| = \frac{2ab}{a+b}$.

The Law of Lever. The Method of the Center of Mass.

Archimedes' Law of Lever.

> "Give me a place to stand on, and I will move the earth."
>
> quoted by Pappus of Alexandria in Synagoge, Book VIII, c. AD 340



Archimedes of Syracuse

Born c. 287 BC

Syracuse, Sicily

Magna Graecia

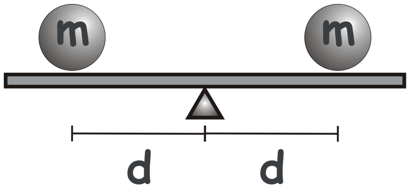
Died c. 212 BC (aged around 75) , Syracuse

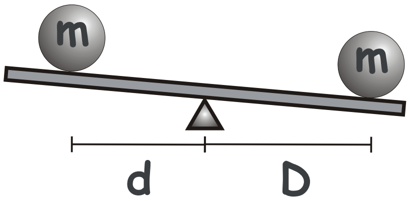
Archimedes of Syracuse generally considered the greatest mathematician of antiquity and one of the greatest of all time. Archimedes anticipated modern calculus and analysis by applying concepts of infinitesimals and the method of exhaustion to derive and rigorously prove a range of geometrical theorems, including the area of a circle, the surface area and volume of a sphere, and the area under a parabola.

He was also one of the first to apply mathematics to physical phenomena, founding hydrostatics and statics, including an explanation of the principle of the lever. He is credited with designing innovative machines, such as his screw pump, compound pulleys, and defensive war machines to protect his native Syracuse from the Roman invasion.

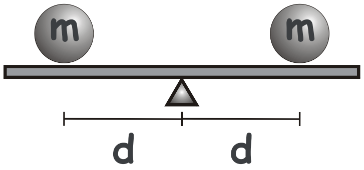
Archimedes derives the Law of Lever from several simple axioms (assumptions), which summarize the everyday experience, in a manner similar to those in Euclidean geometry.

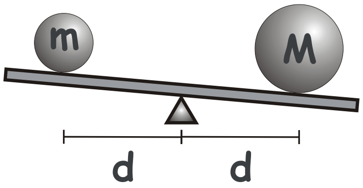
Axiom 1. Equal weights at equal distances from the fulcrum balance. Equal weights at unequal distance from the fulcrum do not balance, but the weight at the greater distance will tilt its end of the lever down.



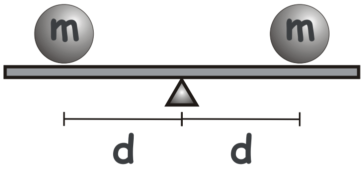


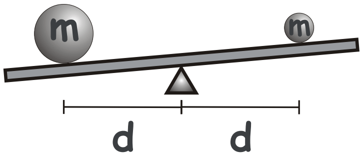
Axiom 2. If, when two weights balance, we add something to one of the weights, they no longer balance. The side holding the weight we increased goes down.





Axiom 3. If, when two weights balance, we take something away from one of them, they no longer balance. The side holding the weight we did not change goes down.





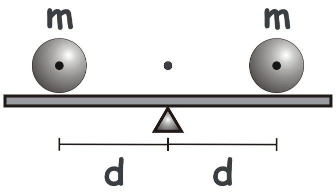
Archimedes then proves the inverse statements as propositions (theorems).

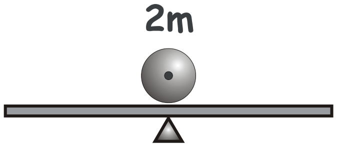
Proposition 1. Weights that balance at equal distances from the fulcrum are equal.

Proposition 2. Unequal weights at equal distances from the fulcrum do not balance, but the side holding the heavier weight goes down.

Proposition 3. Unequal weights balance at unequal distances from the fulcrum, the heavier weight being at the shorter distance.

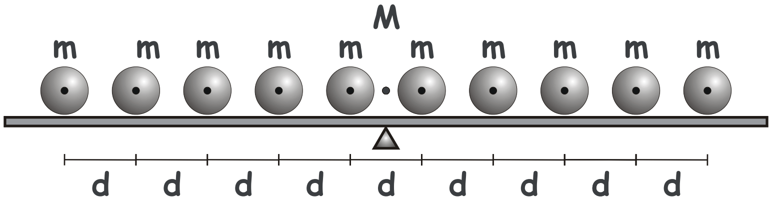
Proposition 4. If two equal weights have different centers of gravity then the center of gravity of the two together is the midpoint of the line segment joining their centers of gravity.





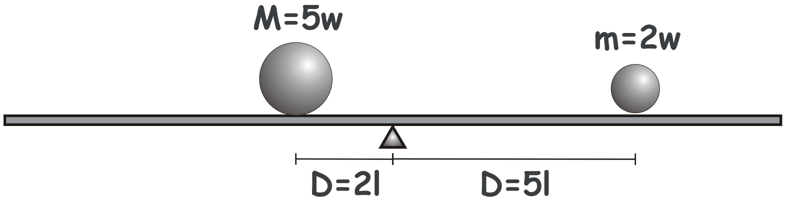
Proposition 4 is just a rephrase of the Axiom 1, where Archimedes tacitly introduces the notion of the **Center of Gravity** (**Center of Mass**). The way to understand the Proposition 4 is to treat the entire weight as if it is located at a single point, its center of gravity. In other words, we can picture each weight (mass) as concentrated in a single point, i. e. as a **Point Mass**. We shall use terms weight and mass interchangeably, assuming that weight is associated with a mass in the homogeneous gravitation field, and therefore is proportional to the mass. The following observation immediately follows from the Proposition 4.

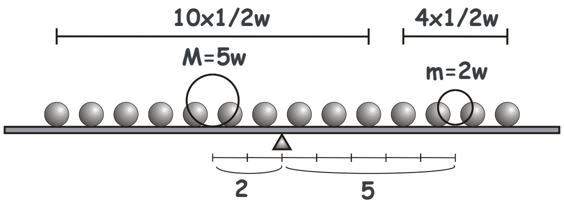
Corollary. If an even number of equal weights have their centers of gravity situated along a straight line such that the distances between the consecutive weights are all equal, then the center of gravity of the entire system is the midpoint of the line segments joining the centers of gravity of the two weights in the middle.



At this point Archimedes proves the Law of Lever, first only for commensurate weights.

Proposition 5. Commensurate weights (masses) balance at distances from the fulcrum, which are inversely proportional to their magnitudes, $\frac{d}{D} = \frac{M}{m}$.





Proof. Let w be the greatest common measure of weights (masses) m and M, $m = pw$, $M = nw$, $\{p, n\} \in N$. Let us split weight M into $2n$ smaller pieces, each of weight $w/2$, and weight m into $2p$ smaller pieces of weight $w/2$. Let us now split the segment connecting M and m into $n + p$ congruent smaller segments, and also mark n such segments on the opposite side of weight M and p such segments on the opposite side of weight m. Let us now place all $2(n + p)$ smaller weights w at the centers of these $2(n + p)$ segments as shown in the Figure. Clearly, since each of the initial weights was split into an even number of equal pieces, which were placed symmetrically around its initial position, the resultant system of smaller weights has the same center of gravity as the original weight. On the other hand, the obtained system of $2(n + p)$ weights $w/2$ has the center of gravity in the middle, at a distance of p segments from the position of weight M and n segments from the position of weight m, as illustrated in the Figure. Therefore, $\frac{D}{d} = \frac{p}{n} = \frac{m}{M}$, which proves the Law of Lever for the commensurate weights. The theorem for the incommensurate weights is then proven by reducing to contradiction.

Theorem (Law of Lever). Incommensurate weights (masses) balance at distances from the fulcrum, which are inversely proportional to their magnitudes,

$$\frac{d}{D} = \frac{M}{m}$$

Proof. Let weights m and M be placed at distances d and D from the fulcrum, respectively, such that the Law of Lever is satisfied, $MD = md$. Assume that the weights nevertheless do not balance, for example, M goes down. Remove a small amount from weight M, turning it into weight M', such that it still goes down, but is now commensurate with m. Now m and M' are commensurate, and $md > M'D$, which means that M' should rise. This contradicts our assumption, so m and M must balance. Note that in the above Archimedes implies a non-trivial fact that a commensurate weight can be found that differs from the given incommensurate weight by an arbitrarily small amount. This means that for any irrational number there exists a rational number, which differs from it as little as we want, i. e. that rational numbers are dense.

Method of the Center of Mass (Mass Points).

Definition. For two point masses, m_A and m_B at points A and B, the center of mass lies at a point C’ on the straight line segment |AB|, such that,

$$\frac{|AC'|}{|C'B|} = \frac{m_B}{m_A}.$$

When finding the center of mass in a system of point masses, one can replace any pair of masses, m_A and m_B, with a single point mass having the total mass $m_A + m_B$, placed at the center of mass of the pair.

The following important properties of the Center of Mass follow immediately.

1. Every system of finite number of point masses has unique center of mass (COM).
2. For two point masses, the COM belongs to the segment connecting these points; its position is determined by the Archimedes lever rule: the point’s mass times the distance from it to the COM is the same for both points.
3. The position of the system’s center of mass does not change if we move any subset of point masses in the system to the center of mass of this subset. In other words, we can replace any number of point masses with a single point mass, whose mass equals the sum of all these masses and which is positioned at their COM.

Ceva’s Theorem: Point Masses.

We select masses, m_A, m_B, and m_C such that the corresponding centers of mass for each pair are at points A’, B’ and C’, respectively. Then,

$$\frac{|AB'|}{|B'C|} \cdot \frac{|CA'|}{|A'B|} \cdot \frac{|BC''|}{|C'A|} = \frac{m_C}{m_A} \cdot \frac{m_B}{m_C} \cdot \frac{m_A}{m_B} = 1.$$