Homework for October 22, 2017.

Algebra.

Review the classwork handout and complete the exercises. Solve the remaining problems from the previous homework (pasted below). Solve the following problems.

1. Consider the progression

   1, 3, 5, 7, …, 993, 995, 997, 999.

   How many terms does it have? Find the sum of these terms.

2. Prove that the sum of n first odd numbers is a perfect square.

   ($1=1^2$, $1+3=2^2$, $1+3+5=3^2$ etc.)

3. Using mathematical induction, prove that

   a. $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
   b. $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$
   c. $\sum_{k=1}^{n} \frac{1}{k^2+k} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$
   d. $\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(n-1)\cdot(n+1)} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$
   e. $\forall n, \exists k, 5^n + 3 = 4k$
   f. $\forall x > -1, \forall n \geq 2, \quad (1+x)^n \geq 1 + nx$

   Recap. In order to prove the equality $A(n) = B(n)$, for any n, using the method of mathematical induction you have to

   a. Prove that $A(1) = B(1)$
   b. Prove that $A(k+1) - A(k) = B(k+1) - B(k)$ (*)
   c. Then from assumption $A(k) = B(k)$ and from equality (*) follows $A(k+1) = B(k+1)$

4. Prove by mathematical induction that for any natural number n, $15^n + 6$ is divisible by 7.

Recap. Problems below are from the previous homework. Some of these problems are solved in the classwork handout posted online. Consider these solutions for an example in solving the remaining problems.

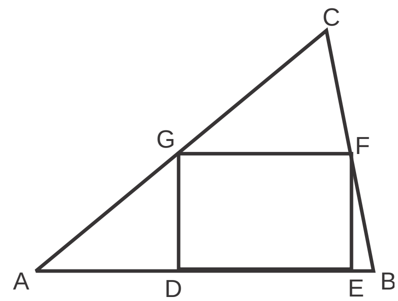
1. Solve the following equations:
   a. $\frac{x-a}{x-b}+\frac{x-b}{x-a}=2.5$
   b. $\sqrt{3x+4}+\sqrt{x-2}=2\sqrt{x}$
   c. $\frac{1}{x^3+2}-\frac{1}{x^3+3}=\frac{1}{12}$
   d. $\frac{1}{x^2}+\frac{1}{(x+2)^2}=\frac{10}{9}$
   e. $\sqrt{x-2}=x-4$
   f. $1+\sqrt{1+x\sqrt{x^2-24}}=x$
   g. $\sqrt{x}+\frac{2x+1}{x+2}=2$
2. Simplify expressions:
   a. $\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x-1}}$
   b. $\frac{\sqrt{\sqrt{\frac{x-1}{x+1}}+\sqrt{\frac{x+1}{x-1}}-2}}{\sqrt{(x+1)^3}-\sqrt{(x-1)^3}}\left(2x+\sqrt{x^2-1}\right)$
   c. $\frac{\sqrt{x-4\sqrt{x-4}+2}}{\sqrt{x+4\sqrt{x-4}-2}}$
3. Prove that:
   a. $\frac{\sqrt{7+4\sqrt{3}}\cdot\sqrt{19-8\sqrt{3}}}{4-\sqrt{3}}-\sqrt{3}=2$
   b. $\sqrt{6a+2\sqrt{9a^2-b^2}}-\sqrt{6a-2\sqrt{9a^2-b^2}}=2\sqrt{3a-b}$

Geometry.

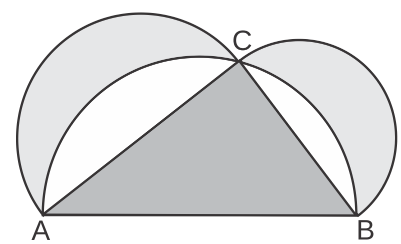
Review the classwork handout. Solve the remaining problems from the previous homework; consider solutions explained in the classwork handout. Try solving the following additional problems. In all the problems, you are only allowed to use theorems we had proven before!

Problems.

1. Find the area and the perimeter of the right triangle such that the lengths of its legs are the roots of the equation, $ax^2 + bx + c = 0$.
2. Rectangle DEFG is inscribed in triangle ABC such that the side DE belongs to the base AB of the triangle, while points F and G belong to sides BC abd CA, respectively. What is the largest area of rectangle DEFG?



3. Prove that the area of the two shaded lunes formed by the semi-circles built on the legs and the hypotenuse of the right triangle ABC equals to the area of the triangle ABC (hint: use the generalized Pythagoras theorem).



4. Using a ruler and a compass, construct a circle inscribed in a given angle, AOC (i.e. tangent to both sides, OA and OC of this angle), and a point M inside this angle.
5. Prove that altitudes of any triangle are the bisectors in another triangle, whose vertices are the feet of these altitudes (hint: prove that the line connecting the feet of two altitudes of a triangle cuts off a triangle similar to it).