October 22, 2017

Algebra.

Principle of Mathematical Induction (continued).

Let $\{P(n)\} = P(1), P(2), P(3), \ldots$ be a sequence of propositions numbered by positive integers, which together constitute the general theorem P. In particular, $P(n)$ can be some formula, or other property of positive integers.

Theorem (Principle of Mathematical Induction).

$$\big(P(1) \wedge (\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1))\big) \Rightarrow (P: \forall n \in \mathbb{N}, P(n)).$$

Proof. Assume the opposite. Recalling that, $\sim(Q \Rightarrow P) \Leftrightarrow (Q \wedge \sim P)$, we write, the negation of the above statement as,

$$\big(P(1) \wedge (\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1))\big) \wedge \sim(P: \forall n \in \mathbb{N}, P(n)),$$

Or,

$$\big(P(1) \wedge (\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1))\big) \wedge (\exists n \in \mathbb{N}, \sim P(n)).$$

Now, using the "principle of smallest integer" we arrive at a contradiction,

$$\Big(\exists r \in \mathbb{N}, \big(P(r-1) \wedge \sim\big(P(r)\big)\big)\Big) \Leftrightarrow \sim(\forall r \in \mathbb{N}, P(r) \Rightarrow P(r+1)). \text{ ¤}$$

Example 1. Prove that the sum of the n first odd positive integers is n^2,

i.e., $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

Solution. Let $S(n) = 1 + 3 + 5 + \cdots + (2n-1)$.

We want to prove by induction that for every positive integer n, $S(n) = n^2$.

(1) Verify Base Case. For $n = 1$, we have $S(1) = 1 = 1^2$, so the property holds for $n = 1$.

(2) Inductive Step. Assume (Induction Hypothesis) that the property is true for a positive integer n, i.e.: $S(n) = n^2$. We must prove that it is also true for

$n + 1$, i.e., $S(n+1) = (n+1)^2$, i. e., $\{S(n) = n^2\} \Rightarrow \{S(n+1) = (n+1)^2\}$. In fact, we can verify this explicitly,

$$S(n+1) = 1 + 3 + 5 + \cdots + (2n-1) + (2n+1) = S(n) + (2n+1).$$

But, by induction hypothesis, $S(n) = n^2$. Hence,

$$S(n+1) = n^2 + (2n+1) = (n+1)^2.$$

This completes the induction and shows that the property is true for all positive integers. ¤

Arithmetic and geometric mean inequality: Proof by induction.

The **arithmetic mean** of n numbers, $\{a_1, a_2, \dots, a_n\}$, is, by definition,

$$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{1}{n}\sum_{i=1}^{n} a_i \qquad (1)$$

The **geometric mean** of n non-negative numbers, $\{a_n \geq 0\}$, is, by definition,

$$G_n = \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} = \sqrt[n]{\prod_{i=1}^{n} a_i} \qquad (2)$$

Theorem. For any set of n non-negative numbers, the arithmetic mean is not smaller than the geometric mean,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} \qquad (3)$$

The standard proof of this fact by mathematical induction is given below.

Induction basis. For $n = 1$ the statement is a true equality. We can also easily prove that it holds for $n = 2$. Indeed, $(a_1 + a_2)^2 - 4a_1a_2 = (a_1 - a_2)^2 \geq 0$ $\Rightarrow a_1 + a_2 \geq 2\sqrt{a_1 a_2}$.

Induction hypothesis. Suppose the inequality holds for any set of n non-negative numbers, $\{a_1, a_2, \dots, a_n\}$.

Induction step. We have to prove that the inequality then also holds for any set of $n + 1$ non-negative numbers, $\{a_1, a_2, \dots, a_{n+1}\}$.

Proof. If $a_1 = a_2 = \cdots = a_n = a_{n+1}$, then the equality, $A_{n+1} = G_{n+1}$, obviously holds. If not all numbers are equal, then there is the smallest (smaller than the mean) and the largest (larger than the mean). Let these be $a_{n+1} < A_{n+1}$, and $a_n > A_{n+1}$. Consider new sequence of n non-negative numbers, $\{a_1, a_2, \dots, a_{n-1}, a_n + a_{n+1} - A_{n+1}\}$. The arithmetic mean for these n numbers is still equal to A_{n+1},

$$\frac{a_1+a_2+\cdots+a_{n-1}+a_n+a_{n+1}-A_{n+1}}{n} = \frac{n+1}{n}A_{n+1} - \frac{1}{n}A_{n+1} = A_{n+1} \qquad (4)$$

Therefore, by induction hypothesis,

$$(A_{n+1})^n \geq a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1} \cdot (a_n + a_{n+1} - A_{n+1}) \qquad (5)$$

$$(A_{n+1})^{n+1} \geq a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1} \cdot (a_n + a_{n+1} - A_{n+1}) \cdot A_{n+1} \qquad (6)$$

Wherein, using $a_{n+1} < A_{n+1}$ and $a_n > A_{n+1}$, as assumed above, we get $(a_n - A_{n+1})(A_{n+1} - a_{n+1}) > 0$, or, $a_n a_{n+1} < (a_n + a_{n+1} - A_{n+1})A_{n+1}$, so we could substitute the last two terms in the product with $a_n \cdot a_{n+1}$, while keeping the inequality. This completes the proof. ¤

Newton's binomial.

The **Newton's binomial** is an expression representing the simplest n-th degree factorized polynomial of two variables, $P_n(x, y) = (x + y)^n$ in the form of the polynomial summation (i.e. expanding the brackets),

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + \binom{n}{n-1}x\,y^{n-1} + \binom{n}{n}y^n, \qquad (1a)$$

$$(x+y)^n = C_n^0x^n + C_n^1x^{n-1}y + C_n^2x^{n-2}y^2 + \cdots + C_n^kx^{n-k}y^k + \cdots + C_n^{n-1}x\,y^{n-1} + C_n^ny^n. \qquad (1b)$$

For $n = 1, 2, 3, \ldots$, these are familiar expressions,

$$(x+y) = x + y,$$

$$(x+y)^2 = x^2 + 2xy + y^2,$$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$,

etc.

The Newton's binomial formula could be established either by directly expanding the brackets, or proven using the mathematical induction.

Exercise. Prove the Newton's binomial using the mathematical induction.

Induction basis. For $n = 1$ the statement is a true equality, $(x+y)^1 = C_1^0 x + C_1^1 y$. We can also easily prove that it holds for $n = 2$. Indeed, $(x+y)^2 = C_2^0 x^2 + C_2^1 xy + C_2^2 y^2$.

Induction hypothesis. Suppose the equality holds for some $n \in N$, that is,

$$(x+y)^n = C_n^0 x^n + C_n^1 x^{n-1} y + C_n^2 x^{n-2} y^2 + \cdots + C_n^k x^{n-k} y^k + \cdots + C_n^{n-1} x\, y^{n-1} + C_n^n y^n$$

Induction step. We have to prove that it then also holds for the next integer, $n+1$,

$$(x+y)^{n+1} = C_{n+1}^0 x^{n+1} + C_{n+1}^1 x^n y + C_{n+1}^2 x^{n-1} y^2 + \cdots + C_{n+1}^k x^{n+1-k} y^k + \cdots + C_{n+1}^n x\, y^n + C_{n+1}^{n+1} y^{n+1}$$

Proof. $(x+y)^{n+1} = (x+y)^n (x+y) =$

$$(C_n^0 x^n + C_n^1 x^{n-1} y + C_n^2 x^{n-2} y^2 + \cdots + C_n^k x^{n-k} y^k + \cdots + C_n^{n-1} x\, y^{n-1} + C_n^n y^n)(x+y) =$$

$$C_n^0 x^{n+1} + C_n^1 x^n y + C_n^2 x^{n-1} y^2 + \cdots + C_n^k x^{n-k+1} y^k + \cdots + C_n^{n-1} x^2\, y^{n-1} + C_n^n x y^n + C_n^0 x^n y + C_n^1 x^{n-1} y^2 + C_n^2 x^{n-2} y^3 + \cdots + C_n^k x^{n-k} y^{k+1} + \cdots + C_n^{n-1} x\, y^n + C_n^n y^{n+1} =$$

$$C_n^0 x^{n+1} + (C_n^1 + C_n^0) x^n y + (C_n^2 + C_n^1) x^{n-1} y^2 + \cdots + (C_n^k + C_n^{k-1}) x^{n-k+1} y^k + \cdots + (C_n^n + C_n^{n-1}) x\, y^n + C_n^n y^{n+1} =$$

$$C_{n+1}^0 x^{n+1} + C_{n+1}^1 x^n y + C_{n+1}^2 x^{n-1} y^2 + \cdots + C_{n+1}^k x^{n+1-k} y^k + \cdots + C_{n+1}^n x\, y^n + C_{n+1}^{n+1} y^{n+1},$$

Where we have used the property of binomial coefficients, $C_n^k + C_n^{k-1} = C_{n+1}^k$.
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Properties of binomial coefficients

Binomial coefficients are defined by

$$C_n^k = {}_kC_n = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Binomial coefficients have clear and important combinatorial meaning.

- There are $\binom{n}{k}$ ways to choose k elements from a set of n elements.
- There are $\binom{n+k-1}{k}$ ways to choose k elements from a set of n if repetitions are allowed.
- There are $\binom{n+k}{k}$ strings containing k ones and n zeros.
- There are $\binom{n+1}{k}$ strings consisting of k ones and n zeros such that no two ones are adjacent.

They satisfy the following identities,

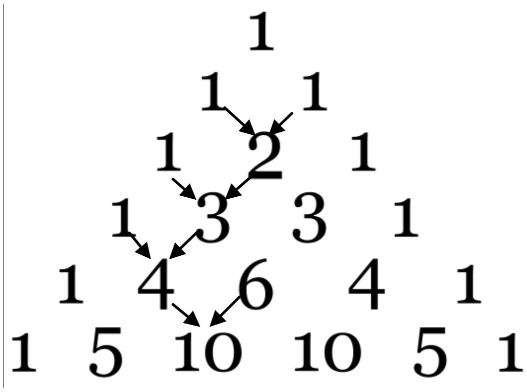
$$C_{n+1}^{k+1} = C_n^k + C_n^{k+1} \Leftrightarrow \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

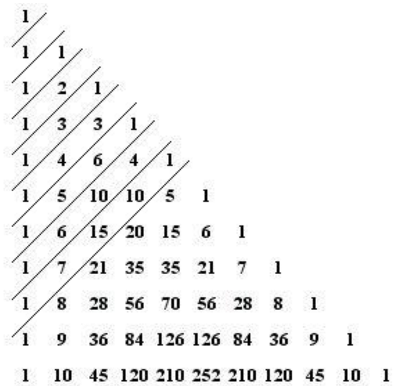
$$C_{n+1}^{k} = C_n^k + C_n^{k-1} \Leftrightarrow \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$\sum_{k=0}^{n} C_n^k = \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Patterns in the Pascal triangle

$C_n^k = C_{n-1}^{k-1} + C_{n-1}^{k}$	Fibonacci numbers (sum of the "shallow" diagonals:





Exercise. Find the sum of the top n rows in the Pascal triangle, $\sum_{m=0}^{n}\left(\sum_{k=0}^{m} C_m^k\right) = 2^{n+1} - 1$.

<u>**Review of selected homework problems**</u>**.**

Problem 4. Using mathematical induction, prove that

a. $P_n: \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Solution.

<u>Basis</u>: $P_1: \sum_{k=1}^{1} k^2 = 1 = \frac{1\cdot(1+1)\cdot(2\cdot1+1)}{6}$

<u>Induction</u>: $P_n \Rightarrow P_{n+1}$, where $P_{n+1}: \sum_{k=1}^{n+1} k^2 = 1^2 + 2^2 + 3^2 + \cdots + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$

<u>Proof</u>:
$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)}{6}(n(2n+1) + 6n + 6) = \frac{(2n+1)(2n^2+7n+6)}{3} = \frac{(n+1)(n+2)(2n+3)}{6}$,

where we used the induction hypothesis, P_n, to replace the sum of the first n terms with a formula given by P_n. ¤

b. $P_n: \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Solution.

Basis: $P_1 : \sum_{k=1}^{1} k^3 = 1 = \left[\frac{1(1+1)}{2}\right]^2$

Induction: $P_n \Rightarrow P_{n+1}$, where $P_{n+1} : \sum_{k=1}^{n+1} k^3 = 1^3 + 2^3 + 3^3 + \cdots + (n+1)^3 = \left[\frac{(n+1)(n+2)}{2}\right]^2$

Proof: $\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3 = \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 = \left[\frac{(n+1)}{2}\right]^2 (n^2 + 4n + 4) = \left[\frac{(n+1)(n+2)}{2}\right]^2$, where we used the induction hypothesis, P_n, to replace the sum of the first n terms with a formula given by P_n. ¤

c. $P_n : \sum_{k=1}^{n} \frac{1}{k^2+k} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$

Solution.

Basis: $P_1 : \sum_{k=1}^{1} \frac{1}{k^2+k} = \frac{1}{2} = \frac{1}{1+1}$

Induction: $P_n \Rightarrow P_{n+1}$, where $P_{n+1} : \sum_{k=1}^{n+1} \frac{1}{k^2+k} = \frac{n+1}{n+2}$

Proof: $\sum_{k=1}^{n+1} \frac{1}{k^2+k} = \sum_{k=0}^{n} \frac{1}{k^2+k} + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{n+1}{n+2}$. ¤

e. $P_n : \forall n, \exists k, 5^n + 3 = 4k$

Solution.

Basis: $P_1 : n = 1, \exists k, 5^1 + 3 = 8 = 4k \Leftrightarrow k = 2$

Induction: $P_n \Rightarrow P_{n+1}$, where $P_{n+1} : \forall n, \exists q, 5^{n+1} + 3 = 4q$

Proof: $5^{n+1} + 3 = 5 \cdot 5^n + 3 = 5 \cdot (4k - 3) + 3 = 5 \cdot 4k - 12 = 4 \cdot (5k - 3)$.

Where we used the induction hypothesis, P_n, to replace 5^n with a formula, $5^n = 4k - 3$, given by P_n. ¤

e. $P_n : \forall n \geq 2, \forall x > -1,\ \ (1 + x)^n \geq 1 + nx$

Solution.

Basis: P_2: $\forall x > -1, n = 2, \;\; (1 + x)^2 = 1 + 2x + x^2 \geq 1 + 2x$

Induction: $P_n \Rightarrow P_{n+1}$, where P_{n+1}: $\forall n \geq 2, \forall x > -1, \; (1 + x)^{n+1} \geq 1 + (n + 1)x$

Proof: $(1 + x)^{n+1} = (1 + x)\,(1 + x)^n \geq (1 + x)(1 + nx) = 1 + (n + 1)x + x^2 \geq 1 + (n + 1)x$. ¤

Recap: Elements of number theory. Eucleadean algorithm and greatest common divisor.

Theorem 1. $\forall a, b \in \mathbb{Z}, b > 0, \exists q, r \in \mathbb{Z}, 0 \leq r < b: a = bq + r$

Proof. If a is a multiple of b, then $\exists q \in \mathbb{Z}, r = 0 : a = bq = bq + r$. Otherwise, $\exists q \in \mathbb{Z} : bq < a < b(q + 1)$. Then, $\exists r = a - bq \in \mathbb{Z} : 0 < r < b$, which completes the proof.

Definition. A number $d \in \mathbb{Z}$ is a common divisor of two integer numbers $a, b \in \mathbb{Z}$, if $\exists n, m \in \mathbb{Z}: a = nd, b = md$.

A set of all positive common divisors of the two numbers $a, b \in \mathbb{Z}$ is limited because these divisors are smaller than the magnitude of the larger of the two numbers. The greatest of the divisors, d, is called the greatest common divisor (gcd) and denoted $d = (a, b)$.

Theorem 2. $\forall a, b, q, r \in \mathbb{Z}, (a = bq + r) \Rightarrow \big((a, b) = (b, r)\big)$

Proof. Indeed, if d is a common divisor of $a, b \in \mathbb{Z}$, then $\exists n, m \in \mathbb{Z}: a = nd, b = md \Rightarrow r = a - bq = (n - mq)d$. Therefore, d is also a common divisor of b and $r = a - bq$. Conversely, if d' is a common divisor of b and $r = a - bq$, then $\exists n', m' \in \mathbb{Z}: b = m'd, a - bq = n'd \Rightarrow a = (n' + m'q)d'$, so d' is a common divisor of b and a. Hence, the statement of the theorem is valid for any divisor of a, b, and for gcd in particular.

Eucleadean algorithm. In order to find the greatest common divisor $d = (a, b)$, one proceeds iteratively performing successive divisions,

$$a = bq_1 + r_1, (a, b) = (b, r_1)$$

$$b = r_1 q_2 + r_2, (b, r_1) = (r_1, r_2)$$

$$r_1 = r_2 q_3 + r_3, (r_1, r_2) = (r_2, r_3), \ldots$$

$$b > r_1 > r_2 > r_3 > \cdots r_n > 0 \Rightarrow \exists n \leq b, r_n = d = (a, b)$$

The last positive reminder, r_n, in the sequence $\{r_k\}$ is (a, b), the gcd of the numbers a and b. Indeed, the Eucleadean algorithm ensures that

$$(a, b) = (b, r_1) = (r_1, r_2) = \cdots = (r_{n-1}, r_n) = (r_n, 0) = r_n$$

Examples.

a. $(385,105) = (105,70) = (70,35) = (35,0) = 35$
b. $(513,304) = (304,209) = (209,95) = (95,19) = (19,0) = 19$

Corollary. $(d = (a, b)) \Rightarrow (\exists\, k, l \in \mathbb{Z} : d = ka + lb)$

Proof. Consider the sequence of remainders in the Eucleadean algorithm, $r_1 = a - bq_1, r_2 = b - r_1 q_2, r_3 = r_1 - r_2 q_3, \ldots, r_n = r_{n-2} - r_{n-1} q_n$. Indeed, the successive substitution gives, $r_1 = a - bq_1, r_2 = b - (a - bq_1)q_2 = k_2 a + l_2 b$, $r_3 = r_1 - (k_2 a + l_2 b)q_3 = k_3 a + l_3 b, , \ldots, r_n = r_{n-2} - (k_{n-1} a + l_{n-1} b)q_n = k_n a + l_n b = d = (a, b)$.

Exercise. Find the representation $d = ka + lb$ for the pairs (385,105) and (513,304) considered in the above examples.

Continued fraction representation. Using the Eucleadean algorithm, one can develop a continued fraction representation for rational numbers,

$$\frac{a}{b} = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{\cdots}}} \\ \qquad + \cfrac{1}{q_{n-1} + \cfrac{1}{q_n}}$$

This is accomplished by successive substitution, which gives,

$$\frac{a}{b} = q_1 + \frac{r_1}{b} = q_1 + \frac{1}{\frac{b}{r_1}}, \frac{b}{r_1} = q_2 + \frac{r_2}{r_1} = q_2 + \frac{1}{\frac{r_1}{r_2}}, \ldots, \frac{r_{n-1}}{r_n} = q_{n+1}.$$

Exercise. Show the continued fraction representations for $\frac{385}{105}, \frac{513}{304}, \frac{105}{385}, \frac{304}{513}$.