December 3, 2017

Algebra.

Continued fractions. Solutions to some homework problems.

1. **Problem.** Find x, where

a.

$$x = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}$$

Solution. Following similar line of argument as in the last homework, we assume that such an x exists and substitute the infinite expression in the denominator on the right with x. We thus obtain a quadratic equation,

$$x = 1 + \frac{1}{x} \Leftrightarrow x^2 - x - 1 = 0 \Leftrightarrow x = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + 1},$$

We have to choose the positive solution because x is obviously positive, thus we obtain, $x = \frac{1+\sqrt{5}}{2}$. This number is known as the golden ratio. Two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities. This condition leads to the same quadratic equation on the ratio, x. Recalling problem 1 from the last homework, where we have shown that,

$$n\ fractions \begin{cases} \dfrac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}} \\ \qquad \qquad \ldots + \dfrac{1}{1} \end{cases} = \frac{F_n}{F_{n+1}}, \text{ where } \{F_n\} = 1,1,2,3,5,8, \ldots \text{ are}$$

Fibonacci numbers, we observe that the ratio of the two consecutive Fibonacci numbers tends to the golden ratio as n tends to infinity. This is because $\frac{1+\sqrt{5}}{2} - 1 = \frac{2}{1+\sqrt{5}}$.

b. $x = \frac{1}{2+\frac{1}{2+\frac{1}{2+\cdots}}}$

Solution. $x = \frac{1}{2+x} \Leftrightarrow x^2 + 2x - 1 = 0 \Leftrightarrow x = 1 \pm \sqrt{2}$,

Where we again choose the positive solution, $x = 1 + \sqrt{2}$.

2. **Problem.** Using the method of mathematical induction, prove the following equalities,
   a. $\sum_{k=0}^{n} k \cdot k! = (n+1)! - 1$

   Solution. Base: $\sum_{k=0}^{1} k \cdot k! = 0 + 1 = (1+1)! - 1$

   Induction: $\sum_{k=0}^{n+1} k \cdot k! = \sum_{k=0}^{n+1} k \cdot k! + (n+1) \cdot (n+1)! = (n+1)! - 1 + (n+1) \cdot (n+1)! = (n+1)! \cdot (n+2) - 1 = (n+2)! - 1$

   b. $\sum_{k=1}^{n} \frac{1}{k^2+k} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$

   Solution. Base: $\sum_{k=1}^{1} \frac{1}{k^2+k} = \frac{1}{1+1} = \frac{1}{1+1}$

   Induction: $\sum_{k=1}^{n+1} \frac{1}{k^2+k} = \sum_{k=1}^{n} \frac{1}{k^2+k} + \frac{1}{(n+1)^2+(n+1)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{n+1}{n+2}$.

3. **Problem.** Put the sign <, >, or =, in place of … below,

$$\frac{n+1}{2} \ldots \sqrt[n]{n!}$$

   Solution. Consider set of n consecutive integers, $\{1,2,3, \ldots n\}$. Using the inequality between the arithmetic and the geometric mean of these n numbers we obtain,

$$\frac{1+2+\cdots+n}{n} \geq \sqrt[n]{1 \cdot 2 \cdot \ldots \cdot n} \Leftrightarrow \frac{n(n+1)}{2n} = \frac{n+1}{2} \geq \sqrt[n]{n!}$$

4. Using the inclusion-exclusion principle, find how many natural numbers $n < 1000$ are not divisible by 7, 11, and 35.

Solution. For $n \leq 1000$, there are 142 numbers divisible by 7, $|A_7| = 142$, 90 divisible by 11, $|A_{11}| = 90$, 28 divisible by 35, $|A_{35}| = 28$. Also, there are 12 numbers divisible by $7 \cdot 11 = 77$, 28 divisible by 7 and 35 (every one divisible by 35 is also divisible by 7), 2 divisible by 11·35=385, and these two are also divisible by 7, 11 and 35. Hence, the answer is $1000 - |A_7 + A_{11} + A_{35}| = 1000 - (142 + 90 + 28 - 12 - 28 - 2 + 2) = 780$.