Homework for February 25, 2017.

Algebra.

Review the classwork handout and complete the exercises which were not solved in class. Try solving the unsolved problems from the previous homework (some are repeated below) and the following new problems.

1. Try proving the following properties of real numbers and arithmetical operations on them using definition of a real number as the Dedekind section and the validity of these properties for rational numbers.

Ordering and comparison.

1. $\forall\, a, b \in \mathbb{R}$, one and only one of the following relations holds
   - $a = b$
   - $a < b$
   - $a > b$
2. $\forall\, a, b \in \mathbb{R}, \exists c \in \mathbb{R}, (c > a) \wedge (c < b)$, i.e. $a < c < b$
3. Transitivity. $\forall\, a, b, c \in \mathbb{R}, \{(a < b) \wedge (b < c)\} \Rightarrow (a < c)$
4. Archimedean property. $\forall\, a, b \in \mathbb{R}, a > b > 0, \exists n \in \mathbb{N}$, such that $a < nb$

Addition and subtraction.

- $\forall\, a, b \in \mathbb{R}, a + b = b + a$
- $\forall\, a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$
- $\forall\, a \in \mathbb{R}, \exists 0 \in \mathbb{R}, a + 0 = a$
- $\forall\, a \in \mathbb{R}, \exists - a \in \mathbb{R}, a + (-a) = 0$
- $\forall\, a, b \in \mathbb{R}, a - b = a + (-b)$
- $\forall\, a, b, c \in \mathbb{R}, (a < b) \Rightarrow (a + c < b + c)$

2. Show that for the set of real numbers, $\mathbb{R}$, cardinality of the set of all possible subsets is greater than that of a continuum of real numbers itself.
3. Prove that the cardinality of the set of all points on a sphere is the same as that of the set of all points on a circle.

4. Represent $\sqrt{2}$ (and $\sqrt{p}$ for any rational p) by using the continuous fraction,

$$\sqrt{2} = a + \cfrac{c}{b + \cfrac{c}{b + \cfrac{c}{b + \cdots}}}$$

Geometry.

Review the classwork handout. Solve the unsolved problems from the previous homework. Solve the following problems.

Problems.

1. Find the distance between the nearest points of the circles,
   a. $(x-2)^2 + y^2 = 4$ and $x^2 + (y-1)^2 = 9$
   b. $(x+3)^2 + y^2 = 4$ and $x^2 + (y-4)^2 = 9$
   c. $(x-2)^2 + (y+1)^2 = 4$ and $(x+1)^2 + (y-3)^2 = 5$
   d. $(x-a)^2 + y^2 = r_1^2$ and $x^2 + (y-b)^2 = r_2^2$
2. Review derivation of the equation describing an ellipse and derive in a similar way,
   a. Equation of an ellipse, defined as the locus of points P for which the distance to a given point (focus F_2) is a constant fraction of the perpendicular distance to a given line, called the directrix, $|PF_2|/|PD| = e < 1$.
   b. Equation of a hyperbola, defined as the locus of points for which the ratio of the distances to one focus and to a line (called the directrix) is a constant e. However, for a hyperbola it is larger than 1, $|PF_2|/|PD| = e > 1$.
3. Find (describe) set of all points formed by the centers of the circles that are tangent to a given circle of radius r and a line at a distance $d > r$ from its center, O.