Homework for April 15, 2018.

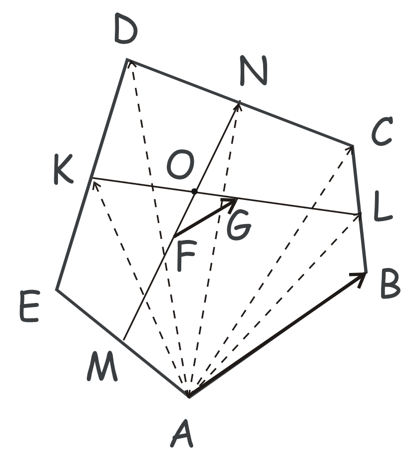
Geometry. Vectors.

Please, complete problems from the previous homework assignment. Review classwork handout on vectors. Solve the following problems.

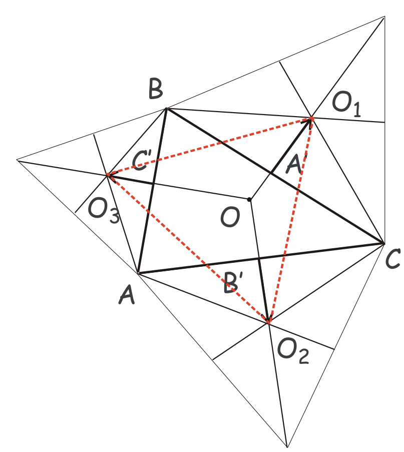
Problems.

1. In a pentagon $ABCDE$, M, K, N and L are the midpoints of the sides AE, ED, DC, and CB, respectively. F and G are the midpoints of thus obtained segments MN and KL (see Figure). Show that the segment FG is parallel to AB and its length is ¼ of that of AB, $|FG| = 1/4|AB|$.

   Hint: use the results of one of the previous problems, expressing the median of a triangle via adjacent sides.



2. Three equilateral triangles are erected externally on the sides of an arbitrary triangle ABC. Show that triangle $O_1O_2O_3$ obtained by connecting the centers of these equilateral triangles is also an equilateral triangle (Napoleon's triangle, see Figure).



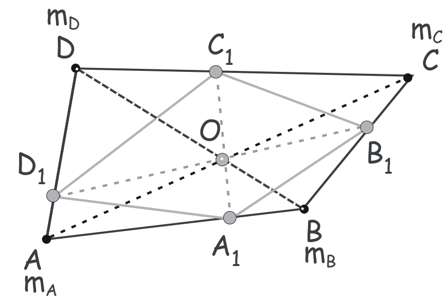
3. If you have not done it yet, solve the following problem from the last homework. Vectors $\overrightarrow{AA'}$, $\overrightarrow{BB'}$ and $\overrightarrow{CC'}$ are represented by the internal bisectors in the triangle ABC, directed from each vertex to the point on the opposite side. Express the sum, $\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'}$ through vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (and the sides of the triangle, $|AB| = c$, $|BC| = a$, $|CA| = b$). For what triangles ABC does this sum equal 0?

4. Let A, B and C be angles of a triangle ABC.

   a. Prove that $\cos A + \cos B + \cos C \le \frac{3}{2}$.

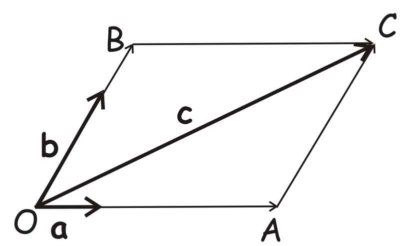
b. *Prove that for any three numbers, m,n,p, $2mn\cos A + 2np\cos B + 2pm\cos C \leq m^2 + n^2 + p^2$

5. *A quadrilateral $A_1B_1C_1D_1$ is inscribed in the quadrilateral ABCD in such a way that diagonals of both quadrilaterals intersect at the same crossing point, O (see Figure). Show that this is possible if $\frac{|AA_1|}{|A_1B|}\frac{|BB_1|}{|B_1C|}\frac{|CC_1|}{|C_1D|}\frac{|DD_1|}{|D_1A|} = 1.$

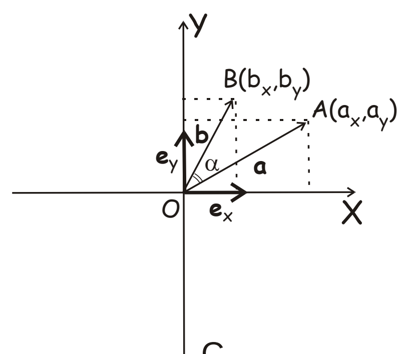


6. Prove that if vectors $\vec{a}$ and $\vec{b}$ satisfy $\|\vec{a}+\vec{b}\| = \|\vec{a}-\vec{b}\|$, then $\vec{a} \perp \vec{b}$.

7. Show that for any two non-collinear vectors $\vec{a}$ and $\vec{b}$ in the plane and any third vector $\vec{c}$ in the plane, there exist one and only one pair of real numbers (x,y) such that $\vec{c}$ can be represented as $\vec{c} = x\vec{a} + y\vec{b}$.



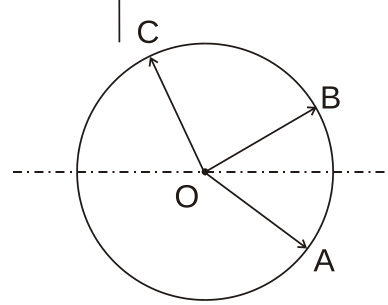
8. Derive the formula for the scalar (dot) product of the two vectors, $\vec{a}(x_a, y_a)$ and $\vec{b}(x_b, y_b)$, $(\vec{a}\cdot\vec{b}) = x_a x_b + y_a y_b$, using their representation via two perpendicular vectors of unit length, $\vec{e}_x$ and $\vec{e}_y$, directed along the X and the Y axis, respectively.



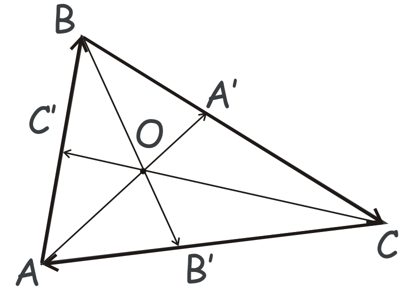
9. Vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are represented by the radial segments directed from the centre O of the circle to points A, b and C on the circle (see Figure). What are the angles AOB, AOC and COB, if

a. $\overrightarrow{OC} = \overrightarrow{OA} - \overrightarrow{OB}$

b. $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$



10. Vectors $\overrightarrow{AA'}$, $\overrightarrow{BB'}$ and $\overrightarrow{CC'}$ are represented by the internal bisectors in the triangle ABC, directed from each vertex to the point on the opposite side (see figure). Express the sum, $\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'}$ through vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (and the sides of the triangle, $|AB| = c, |BC| = a, |CA| = b$). For what triangles ABC does this sum equal 0?



11. Given three vectors, $\vec{a}$, $\vec{b}$ and $\vec{c}$, show that vector $\left(\vec{b}\cdot\vec{c}\right)\vec{a}-(\vec{a}\cdot\vec{c})\vec{b}$ is perpendicular to $\vec{c}$.

12. Given triangle ABC, find the locus of points M such that $\left(\overrightarrow{AB}\cdot\overrightarrow{CM}\right)+\left(\overrightarrow{BC}\cdot\overrightarrow{AM}\right)+\left(\overrightarrow{CA}\cdot\overrightarrow{BM}\right)=0$. Using this finding, prove that three altitudes of the triangle ABC are concurrent (i.e. all three intersect at a common crossing point, the orthocenter of the triangle ABC).

13. Let O be the circumcenter (a center of the circle circumscribed around) and H be the orthocenter (the intersection point of the three altitudes) of a triangle ABC. Prove, that $\overrightarrow{HA}+\overrightarrow{HB}+\overrightarrow{HC}=2\overrightarrow{HO}$.