Homework for April 22, 2018. Math 9

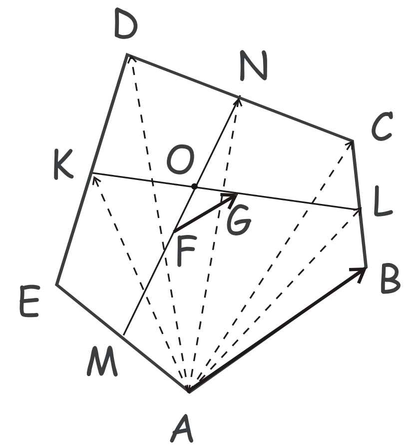
Geometry. Vectors.

Review classwork handout on vectors. Please, complete the problems from the previous homework assignment, which are repeated below.

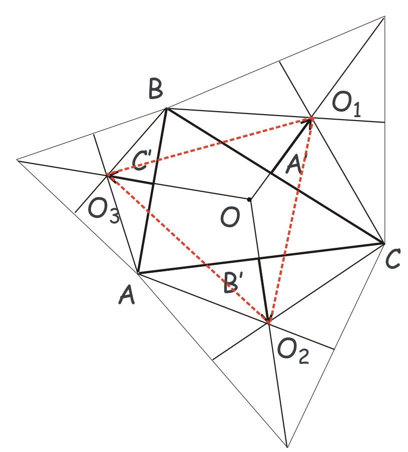
Problems.

1. In a pentagon $ABCDE$, M, K, N and L are the midpoints of the sides AE, ED, DC, and CB, respectively. F and G are the midpoints of thus obtained segments MN and KL (see Figure). Show that the segment FG is parallel to AB and its length is ¼ of that of AB, $|FG| = 1/4|AB|$.

   Hint: use the results of one of the previous problems, expressing the median of a triangle via adjacent sides.



2. Three equilateral triangles are erected externally on the sides of an arbitrary triangle ABC. Show that triangle $O_1O_2O_3$ obtained by connecting the centers of these equilateral triangles is also an equilateral triangle (Napoleon's triangle, see Figure).



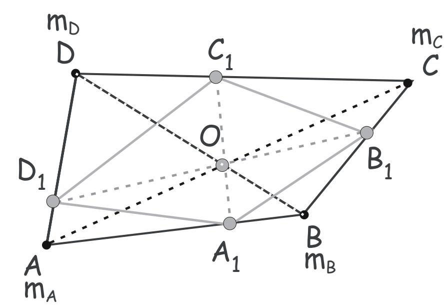
3. If you have not done it yet, solve the following problem from the last homework. Vectors $\overrightarrow{AA'}$, $\overrightarrow{BB'}$ and $\overrightarrow{CC'}$ are represented by the internal bisectors in the triangle ABC, directed from each vertex to the point on the opposite side. Express the sum, $\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'}$ through vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (and the sides of the triangle, $|AB| = c$, $|BC| = a$, $|CA| = b$). For what triangles ABC does this sum equal 0?

4. Let A, B and C be angles of a triangle ABC.

   a. Prove that $\cos A + \cos B + \cos C \leq \frac{3}{2}$.

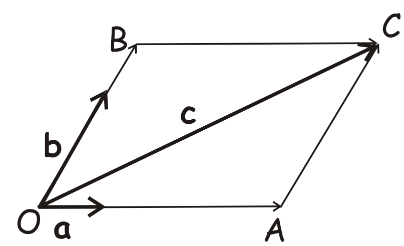
b. *Prove that for any three numbers, m,n,p, $2mn\cos A + 2np\cos B + 2pm\cos C \leq m^2 + n^2 + p^2$

5. *A quadrilateral $A_1B_1C_1D_1$ is inscribed in the quadrilateral $ABCD$ in such a way that diagonals of both quadrilaterals intersect at the same crossing point, O (see Figure). Show that this is possible if $\frac{|AA_1|}{|A_1B|}\frac{|BB_1|}{|B_1C|}\frac{|CC_1|}{|C_1D|}\frac{|DD_1|}{|D_1A|} = 1$.

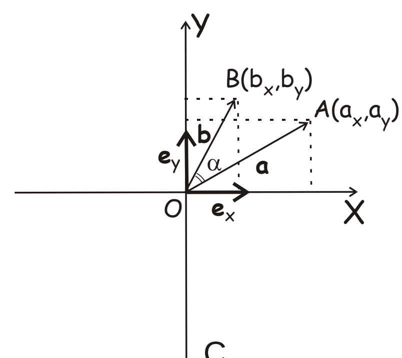


6. Prove that if vectors $\vec{a}$ and $\vec{b}$ satisfy $\|\vec{a}+\vec{b}\| = \|\vec{a}-\vec{b}\|$, then $\vec{a} \perp \vec{b}$.

7. Show that for any two non-collinear vectors $\vec{a}$ and $\vec{b}$ in the plane and any third vector $\vec{c}$ in the plane, there exist one and only one pair of real numbers (x,y) such that $\vec{c}$ can be represented as $\vec{c} = x\vec{a} + y\vec{b}$.



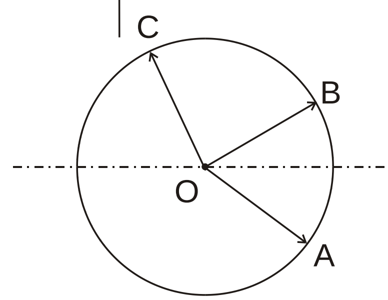
8. Derive the formula for the scalar (dot) product of the two vectors, $\vec{a}(x_a, y_a)$ and $\vec{b}(x_b, y_b)$, $(\vec{a} \cdot \vec{b}) = x_a x_b + y_a y_b$, using their representation via two perpendicular vectors of unit length, $\vec{e}_x$ and $\vec{e}_y$, directed along the X and the Y axis, respectively.



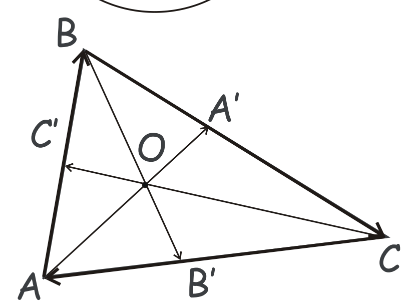
9. Vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are represented by the radial segments directed from the centre O of the circle to points A, b and C on the circle (see Figure). What are the angles AOB, AOC and COB, if

a. $\overrightarrow{OC} = \overrightarrow{OA} - \overrightarrow{OB}$

b. $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$



10. Vectors $\overrightarrow{AA'}$, $\overrightarrow{BB'}$ and $\overrightarrow{CC'}$ are represented by the internal bisectors in the triangle ABC, directed from each vertex to the point on the opposite side (see figure). Express the sum, $\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'}$ through vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (and the sides of the triangle, $|AB| = c, |BC| = a, |CA| = b$). For what triangles ABC does this sum equal 0?



Algebra: polynomials recap.

Review the classwork notes on polynomials, factorization and Vieta theorem. Solve the following problems.

Problems.

1. Write Vieta formulae for the cubic equation, $x^3 + Px^2 + Qx + R = 0$. Let x_1, x_2 and x_3 be the roots of this equation. Find the following combination in terms of P, Q and R,
   a. $(x_1 + x_2 + x_3)^2$
   b. ${x_1}^2 + {x_2}^2 + {x_3}^2$
   c. $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$
   d. $(x_1 + x_2 + x_3)^3$
2. The three real numbers x, y, z, satisfy the equations

$$x + y + z = 6$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{11}{6}$$

$$xy + yz + zx = 11$$

   a. Find a cubic polynomial whose roots are x, y, z
   b. Find x, y, z
3. Find two numbers u, v such that

$$u + v = 6$$

$$uv = 13$$

4. Find three numbers, a, b, c, such that

$$a + b + c = 2$$

$$ab + bc + ca = -7$$

$$abc = -14$$

5. Find all real roots of the following polynomial and factor it.
   a. $x^8 + x^4 + 1$
   b. $x^4 - x^3 + 5x^2 - x - 6$
   c. $x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$
6. Perform the long division, finding the quotient and the remainder, on the following polynomials.
   a. $(x^3 - 3x^2 + 4) \div (x^2 + 1)$
   b. $(x^3 - 3x^2 + 4) \div (x^2 - 1)$