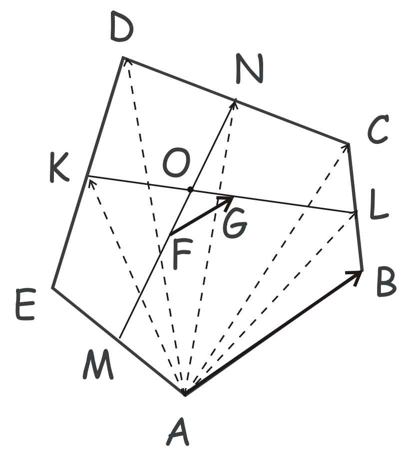
April 15, 2018 Math 9

Geometry.

Solving vector problems.

Problem. In a pentagon $ABCDE$, M, K, N and L are the midpoints of the sides AE, ED, DC, and CB, respectively. F and G are the midpoints of thus obtained segments MN and KL (see Figure). Show that the segment FG is parallel to AB and its length is ¼ of that of AB, $|FG| = 1/4|AB|$.



Solution. Express $\overrightarrow{FG}$ via sides of the pentagon,

$$\overrightarrow{FG} = \frac{1}{2}\overrightarrow{NM} + \frac{1}{2}\overrightarrow{EA} + \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} + \frac{1}{2}\overrightarrow{LK},$$

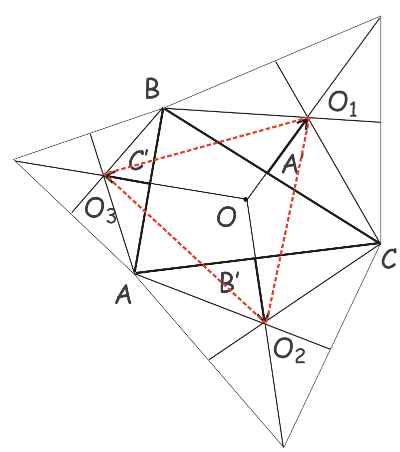
$$\overrightarrow{NM} = \frac{1}{2}\overrightarrow{CD} + \overrightarrow{DE} + \frac{1}{2}\overrightarrow{EA},$$

$$\overrightarrow{LK} = \frac{1}{2}\overrightarrow{BC} + \overrightarrow{CD} + \frac{1}{2}\overrightarrow{DE}.$$

$$\overrightarrow{FG} = \frac{1}{2}\left(\frac{1}{2}\overrightarrow{CD} + \overrightarrow{DE} + \frac{1}{2}\overrightarrow{EA}\right) + \frac{1}{2}\overrightarrow{EA} + \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} + \frac{1}{2}\left(\frac{1}{2}\overrightarrow{BC} + \overrightarrow{CD} + \frac{1}{2}\overrightarrow{DE}\right), \text{ or,}$$
$$\overrightarrow{FG} = \frac{3}{4}\overrightarrow{BC} + \frac{3}{4}\overrightarrow{CD} + \frac{3}{4}\overrightarrow{DE} + \frac{3}{4}\overrightarrow{EA} + \overrightarrow{AB} = \frac{3}{4}\left(\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA}\right) + \frac{1}{4}\overrightarrow{AB}$$

Or, $\overrightarrow{FG} = \frac{1}{4}\overrightarrow{AB}$, since $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA} = 0$.

Problem. Three equilateral triangles are erected externally on the sides of an arbitrary triangle ABC. Show that triangle $O_1O_2O_3$ obtained by connecting the centers of these equilateral triangles is also an equilateral triangle (Napoleon's triangle, see Figure).



Solution. Denote |AB| = c, |BC| = a, |AC| = b. Let us find the side $|O_2O_3|$. Express $\overrightarrow{O_2O_3} = \overrightarrow{AO_3} - \overrightarrow{AO_2}$, or,
$\overrightarrow{O_2O_3} = \frac{1}{2}\overrightarrow{AB} + \overrightarrow{C'O_3} - \frac{1}{2}\overrightarrow{AC} - \overrightarrow{B'O_2}$.

Note, that $\left|\overrightarrow{B'O_2}\right| = b\frac{\sqrt{3}}{6}$, and $\left|\overrightarrow{C'O_3}\right| = c\frac{\sqrt{3}}{6}$. Also, $\left(\overrightarrow{AB}\cdot\overrightarrow{AC}\right) = bc\cos\alpha$,
$\left(\overrightarrow{AB}\cdot\overrightarrow{B'O_2}\right) = \left(\overrightarrow{AC}\cdot\overrightarrow{C'O_3}\right) = bc\frac{\sqrt{3}}{6}\cos(90^\circ+\alpha) = -\frac{1}{2\sqrt{3}}bc\sin\alpha$, and
$\left(\overrightarrow{C'O_3}\cdot\overrightarrow{B'O_2}\right) = \frac{1}{12}bc\cos(180^\circ-\alpha) = -\frac{1}{12}bc\cos\alpha$, where $\alpha = \widehat{BAC}$. Then,

$$\left|\overrightarrow{O_2O_3}\right|^2 = \frac{1}{4}\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{C'O_3}\right|^2 + \frac{1}{4}\left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{B'O_2}\right|^2 - \frac{1}{2}\left(\overrightarrow{AB}\cdot\overrightarrow{AC}\right)$$

$$-\left(\overrightarrow{AB}\cdot\overrightarrow{B'O_2}\right) - \left(\overrightarrow{AC}\cdot\overrightarrow{C'O_3}\right) - 2\left(\overrightarrow{C'O_3}\cdot\overrightarrow{B'O_2}\right),$$ or,

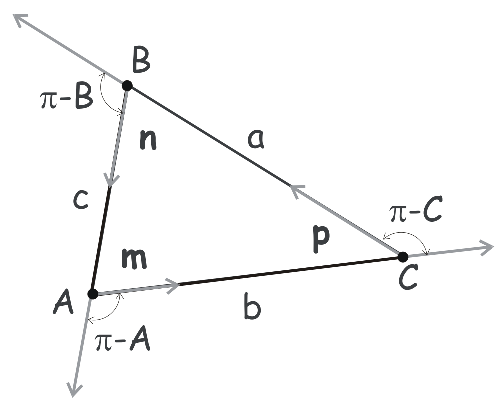
$$\left|\overrightarrow{O_2O_3}\right|^2 = \frac{1}{4}\left(c^2 + \frac{1}{3}c^2 + b^2 + \frac{1}{3}b^2 - 2bc\cos\alpha + \frac{4}{\sqrt{3}}bc\sin\alpha + \frac{2}{3}bc\cos\alpha\right),$$

$$\left|\overrightarrow{O_2O_3}\right|^2 = \frac{1}{3}c^2 + \frac{1}{3}b^2 - \frac{1}{3}bc\cos\alpha + \frac{1}{\sqrt{3}}bc\sin\alpha.$$

Now, using the Law of cosines, $2bc\cos\alpha = b^2 + c^2 - a^2$, and the Law of sines, $\sin\alpha = \frac{a}{2R}$, where R is the radius of the circumcircle, we obtain $\left|\overrightarrow{O_2O_3}\right|^2 = \frac{1}{6}a^2 + \frac{1}{6}b^2 + \frac{1}{6}c^2 + \frac{abc}{2\sqrt{3}R}$. Obviously, the same expression holds for the sides $|O_1O_3|$ and $|O_1O_2|$. Hence, triangle $O_1O_2O_3$ is equilateral.

Problem. Let A, B and C be angles of a triangle ABC.

a. Prove that $\cos A + \cos B + \cos C \le \frac{3}{2}$.
b. *Prove that for any three numbers, m,n,p, $2mn\cos A + 2np\cos B + 2pm\cos C \le m^2 + n^2 + p^2$



Solution. Let vectors $\vec{m},\vec{n},\vec{p}$ be parallel to $\overrightarrow{AC}$, $\overrightarrow{BA}$ and $\overrightarrow{CB}$, respectively, as in the Figure. Then,

$$(\vec{m}+\vec{n}+\vec{p})^2 = m^2 + n^2 + p^2 - 2mn\cos A - 2np\cos B - 2mp\cos C$$

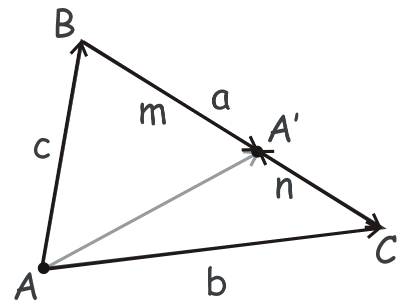
wherefrom immediately follows that,

$$2mn\cos A + 2np\cos B + 2pm\cos C \le m^2 + n^2 + p^2.$$

The statement in part (a) follows from the above for $m = n = p = 1$.

Problem. Point A' divides the side BC of the triangle ABC into two segments, BA' and $A'C$, whose lengths have the ratio $|BA'|:|A'C| = m:n$. Express vector $\overrightarrow{AA'}$ via vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Find the length of the Cevian AA' if the sides of the triangle are $|AB| = c$, $|BC| = a$, and $|AC| = b$.

Solution. It is clear from the Figure, that $\overrightarrow{BA'} = \frac{m}{n}\overrightarrow{A'C} = \frac{m}{m+n}\overrightarrow{BC}$, and $\overrightarrow{CA'} = \frac{n}{m+n}\overrightarrow{CB} = \frac{n}{m+n}\left(\overrightarrow{AB} - \overrightarrow{AC}\right)$. Therefore,



$\overrightarrow{AA'} = \overrightarrow{AC} + \overrightarrow{CA'} = \overrightarrow{AC} + \frac{n}{m+n}\left(\overrightarrow{AB} - \overrightarrow{AC}\right) = \frac{n}{m+n}\overrightarrow{AB} + \frac{m}{m+n}\overrightarrow{AC}$.

Or, we can obtain the same result as

$\overrightarrow{AA'} = \overrightarrow{AB} + \overrightarrow{BA'} = \overrightarrow{AB} + \frac{m}{m+n}\left(\overrightarrow{AC} - \overrightarrow{AB}\right) = \frac{n}{m+n}\overrightarrow{AB} + \frac{m}{m+n}\overrightarrow{AC}$.

For the length of the segment AA' we have,

$|AA'|^2 = \overrightarrow{AA'}^2 = \left(\frac{n}{m+n}\overrightarrow{AB} + \frac{m}{m+n}\overrightarrow{AC}\right)^2 = \frac{n^2c^2+m^2b^2+(nm)2bc\cos\widehat{BAC}}{(m+n)^2}$. Using the Law of cosines, we write $2bc\cos\widehat{BAC} = b^2 + c^2 - a^2$, and obtain the final result,

$|AA'|^2 = \frac{(n^2+nm)c^2+(m^2+nm)b^2-(mn)a^2}{(m+n)^2} = \frac{mb^2+nc^2}{m+n} - \frac{mna^2}{(m+n)^2}$.

Or, equivalently, $(m+n)|BB'|^2 = mb^2 + nc^2 - \frac{mna^2}{m+n}$.

Substituting $m + n = a$, we obtain the Stewart's theorem (Coxeter, Greitzer, exercise 4 on p. 6).

<u>If AA' is a median</u>, then $|BA'|:|A'C| = 1:1$, i.e. $m = n = 1$, and we have, $\overrightarrow{AA'} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}$, $|AA'|^2 = \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{4}a^2$ (AA' is a median).

<u>If AA' is a bisector</u>, $|BA'|:|A'C| = c:b$, i.e. $m = c, n = b$, and we obtain $\overrightarrow{AA'} = \frac{b}{b+c}\overrightarrow{AB} + \frac{c}{b+c}\overrightarrow{AC}$, as well as $|AA'|^2 = \frac{b^2c+c^2b}{b+c} - \frac{bca^2}{(b+c)^2} = bc\left(1 - \frac{a^2}{(b+c)^2}\right)$ (AA' is a bisector).