May 13, 2018 Math 9

Algebra.

Homework review.

1. **Problem .** Find the roots of the following cubic equations by heuristic guess-and-check factorization, and using the Cardano-Tartaglia formula. Reconcile the two results.

    a. $z^3 - 7z + 6 = 0$

    b. $z^3 - 21z - 20 = 0$

    c. $z^3 - 3z = 0$

    d. $z^3 + 3z = 0$

    e. $z^3 - \frac{3}{4}z + \frac{1}{4} = 0$

Solution.

a. Find the roots of the cubic equation,

$$z^3 - 7z - 6 = 0$$

Using Cardano-Tartaglia formula, and by heuristic guess-and-check factorization. Reconcile the two results.

First, we note that $z^3 - 7z - 6 = (z-3)(z+2)(z+1)$, and so the roots are $z = -1, -2, 3$. On the other hand, using the Cardano-Tartaglia formula for the cubic equation, $z^3 + pz + q = 0$,

$z = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$. Substituting $p = -7, q = -6$, we obtain,

$$z = \sqrt[3]{3 - \sqrt{9 - \frac{7^3}{27}}} + \sqrt[3]{3 + \sqrt{9 - \frac{7^3}{27}}} = \sqrt[3]{3 - \frac{10i\sqrt{3}}{9}} + \sqrt[3]{3 + \frac{10i\sqrt{3}}{9}}$$

To show that these solutions are equivalent to the above, we need to compute the cube-roots. Let us write, $w = \sqrt[3]{3 + \frac{10i\sqrt{3}}{9}} = a + bi$. Then,

$3 + \frac{10i\sqrt{3}}{9} = w^3 = (a + bi)^3 = a^3 + 3a^2 ib + 3a(ib)^2 + (ib)^3 = a^3 - 3ab^2 + i(3a^2 b - b^3)$, or,

$$\begin{cases} a^3 - 3ab^2 = 3 \\ 3a^2 b - b^3 = \frac{10\sqrt{3}}{9} = \frac{10}{3\sqrt{3}} \end{cases} \Leftrightarrow \begin{cases} a(a^2 - 3b^2) = 3 \\ b(3a^2 - b^2) = \frac{10}{3\sqrt{3}} \end{cases} \Leftrightarrow \begin{cases} a(a^2 - c^2) = 3 \\ c(9a^2 - c^2) = 10 \end{cases}, b = \frac{c}{\sqrt{3}}.$$

Dividing the two equations we obtain, $\frac{c}{a}\frac{9a^2 - c^2}{a^2 - c^2} = \frac{10}{3} \Leftrightarrow \frac{1}{t}\frac{9t^2 - 1}{t^2 - 1} = \frac{10}{3}, t = \frac{a}{c}$, or,

$$3(9t^2 - 1) = 10(t^3 - t) \Leftrightarrow 10t^3 - 27t^2 - 10t + 3 = 0$$

We again obtained a cubic equation. One root, $t = 3$, is easily guessed, and can be factored out, and the remaining roots can be solved out, or guessed,

$$10t^3 - 27t^2 - 10t + 3 = (t - 3)(10t^2 + 3t - 1) = (t - 3)\left(t + \frac{1}{2}\right)(10t - 2)$$

Substituting the obtained $t = 3, -\frac{1}{2}, \frac{1}{5}$ in the above equations, we obtain,

$a(a^2 - c^2) = t(t^2 - 1)c^3 = 3 \Leftrightarrow t(t^2 - 1)b^3 = \frac{1}{\sqrt{3}} \Leftrightarrow b = \frac{1}{\sqrt[3]{\sqrt{3}t(t^2 - 1)}}$, so

$b = \frac{1}{2\sqrt{3}}, \frac{2}{\sqrt{3}}, -\frac{5}{2\sqrt{3}}$, and $a = tc = tb\sqrt{3} = \frac{3}{2}, -1, -\frac{1}{2}$. We thus found all three cubic roots of $3 \pm \frac{10i\sqrt{3}}{9}$, which are, $\frac{3}{2} \pm \frac{i}{2\sqrt{3}}, -1 \pm \frac{2i}{\sqrt{3}}, -\frac{1}{2} \mp \frac{5i}{2\sqrt{3}}$, and the roots given by the Cardano formula above are 3, -2 and -1, as we have found by guess and check.

c and d are considered more generally above, as a special case, $q = 0$. Consider e. Here $p = -\frac{3}{4}, q = \frac{1}{4}, \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = \sqrt{\frac{1}{4^3} - \frac{3^3}{4^3 \cdot 27}} = 0$, and the Cardano formula gives 3 real roots,

$$y_{0,1,2} = -\frac{1}{2}\left(\sqrt[3]{1} + \frac{1}{\sqrt[3]{1}}\right) = -\frac{1}{2}\left(\sqrt[3]{1} + \overline{\sqrt[3]{1}}\right) = \left\{-1, \frac{1}{2}, \frac{1}{2}\right\}.$$

2. **Problem .** Find all roots of the polynomial, $1 + z + z^2 + z^3 + \cdots + z^n$.

Solution. We note that this polynomial is just a geometric series, so,

$$(1-z)(1+z+z^2+z^3+\cdots+z^n)=1-z^{n+1}$$

Therefore, roots of the polynomial $1+z+z^2+z^3+\cdots+z^n$ coincide with the roots of the polynomial, $1-z^{n+1}$, except for $z=1$. In other words, the roots of the polynomial in question are given by n values of the root-$(n+1)$ of 1 different from 1.

3. **Trigonometric series**

$$S_1=\cos x+\cos 2x+\cdots+\cos nx=?$$

$$S_2=\sin x+\sin 2x+\cdots+\sin nx=?$$

Solution 1. Using the de Moivre formula and the sum of the geometric progression,

$$\begin{aligned}S_1+iS_2&=\cos x+i\sin x+\cos 2x+i\sin 2x\ldots+\cos nx+i\sin nx\\&=(\cos x+i\sin x)+(\cos x+i\sin x)^2+\cdots+(\cos x+i\sin x)^n\\&=\frac{1-(\cos x+i\sin x)^{n+1}}{1-(\cos x+i\sin x)}-1\end{aligned}$$

Now we have to separate the real and the imaginary part. For that, we multiply the denominator and the numerator with the complex conjugate of the denominator,

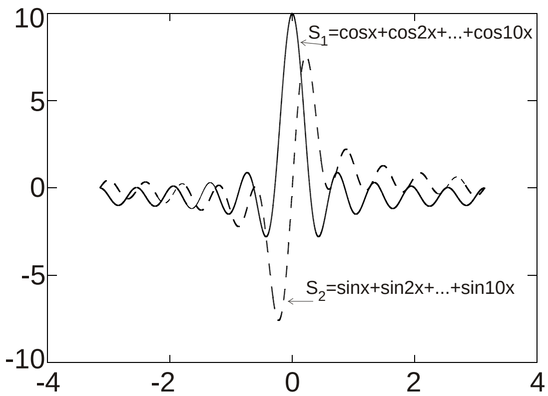
$$S_1+iS_2+1=\frac{(1-(\cos x+i\sin x)^{n+1})\big(1-(\cos x-i\sin x)\big)}{\big(1-(\cos x+i\sin x)\big)\big(1-(\cos x-i\sin x)\big)}=$$
$$\frac{(1-(\cos x+i\sin x)^{n+1})(1-(\cos x+i\sin x)^{-1})}{\big((1-\cos x)+i\sin x\big)\big((1-\cos x)-i\sin x\big)}=$$
$$\frac{1-(\cos x+i\sin x)^{n+1}-(\cos x-i\sin x)+(\cos x+i\sin x)^n}{(1-\cos x)^2-(i\sin x)^2}=$$
$$\frac{1-(\cos x+i\sin x)^{n+1}-(\cos x-i\sin x)+(\cos x+i\sin x)^n}{1-2\cos x+\cos^2 x+\sin^2 x}=$$
$$\frac{1-\cos x+(\cos nx-\cos(n+1)x)+i\big(\sin x+(\sin nx-\sin(n+1)x)\big)}{2-2\cos x}=$$
$$\frac{2\sin^2\frac{x}{2}+2\sin\frac{x}{2}\sin\left(n+\frac{1}{2}\right)x+i\left(2\sin\frac{x}{2}\cos\frac{x}{2}-2\sin\frac{x}{2}\cos\left(n+\frac{1}{2}\right)x\right)}{4\sin^2\frac{x}{2}}=\frac{\sin\frac{x}{2}+\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}+$$
$$i\,\frac{\cos\frac{x}{2}-\cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}.$$

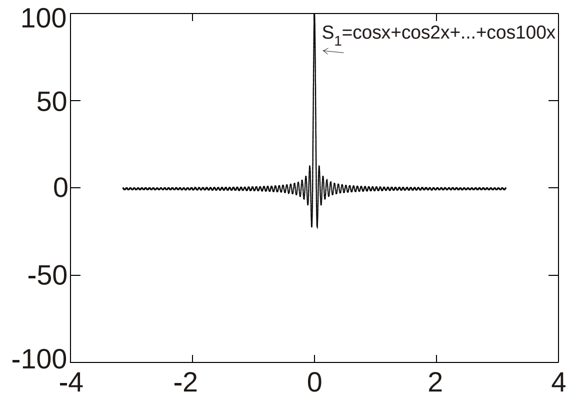
From here, we obtain the trigonometric sums,

$$S_1 = \frac{\sin\frac{x}{2} + \sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} - 1 = \frac{\sin\left(n+\frac{1}{2}\right)x - \sin\frac{x}{2}}{2\sin\frac{x}{2}} = \frac{\sin n\frac{x}{2}}{\sin\frac{x}{2}}\cos\frac{(n+1)x}{2}$$

$$S_2 = \frac{\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} = \frac{\sin n\frac{x}{2}}{\sin\frac{x}{2}}\sin\frac{(n+1)x}{2}$$

It is interesting to look at functions $S_1(x)$ and $S_2(x)$.





Behavior of $S_1(x)$ is intuitively clear. For $x = 0$, all terms in the sum are equal to 1, and the sum equals to the number of terms, $S_1(0) = n$, while for $x \neq 0$ it consists of a large number of positive and negative terms, which tend to cancel each other.

Solution 2. A different and perhaps easier way of summing the above trigonometric series is by adding the expression for S_1, or S_2, rearranged from back to front, to itself , as we did when summing the arithmetic series,

$$S_1 = \cos x + \cos 2x + \cdots + \cos nx$$

$$S_1 = \cos nx + \cos(n-1)x + \cdots + \cos x$$

Wherefrom,

$S_1 = \frac{1}{2}\Big((\cos x + \cos nx) + (\cos 2x + \cos(n-1)x) + \cdots + (\cos nx + \cos x)\Big) =$
$\cos\frac{(n+1)x}{2}\left(\cos(n-1)\frac{x}{2} + \cos(n-3)\frac{x}{2} + \cdots + \cos(n-1)\frac{x}{2}\right) =$
$\cos\frac{(n+1)x}{2}\frac{\left(\sin\frac{x}{2}\cos(n-1)\frac{x}{2}+\sin\frac{x}{2}\cos(n-3)\frac{x}{2}+\cdots+\sin\frac{x}{2}\cos(n-1)\frac{x}{2}\right)}{\sin\frac{x}{2}} =$
$\cos\frac{(n+1)x}{2}\frac{\frac{1}{2}\left(\sin\frac{nx}{2}-\sin\frac{(n-2)x}{2}+\sin\frac{(n-2)x}{2}-\sin\frac{(n-4)x}{2}+\cdots+\sin\frac{nx}{2}\right)}{\sin\frac{x}{2}} = \cos\frac{(n+1)x}{2}\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}.$

Solution 3. Yet even easier way of summing the trigonometric series is by multiplying and dividing it with $\sin\frac{x}{2}$, as we did in the last step above,

$S_1\frac{\sin\frac{x}{2}}{\sin\frac{x}{2}} = \frac{\sin\frac{x}{2}(\cos x+\cos 2x+\cdots+\cos nx)}{\sin\frac{x}{2}} = \frac{\sin\frac{x}{2}\cos x+\sin\frac{x}{2}\cos 2x+\cdots+\sin\frac{x}{2}\cos nx}{\sin\frac{x}{2}} =$
$\frac{\frac{1}{2}\left(-\sin\frac{x}{2}+\sin\frac{3x}{2}-\sin\frac{3x}{2}+\sin\frac{5x}{2}-\sin\frac{5x}{2}+\cdots-\sin\left(n-\frac{1}{2}\right)x+\sin\left(n+\frac{1}{2}\right)x\right)}{\sin\frac{x}{2}} = \frac{\frac{1}{2}\left(-\sin\frac{x}{2}+\sin\left(n+\frac{1}{2}\right)x\right)}{\sin\frac{x}{2}} =$
$\frac{\cos\frac{(n+1)x}{2}\sin\frac{nx}{2}}{\sin\frac{x}{2}}.$

<u>**Recap: Functions and graphs**</u>.

In mathematics, the **graph** of a **function** f is the representation of a **collection of all ordered pairs** $(x, f(x))$. In particular, if x is a real number, graph means the graphical representation of this collection, in the form of a **curve on a Cartesian plane**, together with Cartesian axes, etc. Graphing on a Cartesian plane is sometimes referred to as **curve sketching.** For functions of two variables, the function input x is an ordered pair (x_1, x_2) of real numbers, the graph is the collection of all ordered triples $\big((x_1, x_2), f(x_1, x_2)\big)$, and its graphical representation is a surface in the three-dimensional space.

The graph of a function on real numbers is identical to the graphic representation of the function. For general functions, the graphic representation cannot be applied and the formal definition of the graph of a function suits the need of mathematical statements, e.g., the closed graph theorem in functional analysis. The concept of the graph of a function is generalized to the graph of a relation. Note that although a function is always identified with its graph, strictly speaking they are not the same.

To test if a graph of a curve is a function, use the **vertical line test.** To test if the **function is one-to-one**, meaning it has an **inverse function**, use the horizontal line test. If the function has an inverse, <u>the graph of the inverse can be found by reflecting the graph of the original function over the line $y = x$</u>. A curve is a one-to-one function if and only if it is a function and it passes the horizontal line test.

In science, engineering, technology, finance, and other areas, graphs are tools used for many purposes. In the simplest case one variable is plotted as a function of another, typically using rectangular Cartesian axes.

Surjection, injection, bijection.

Let f be a function from set X to set Y: $X \xrightarrow{f} Y$. Consider the equation,

$$y = f(x)$$

If such an equation always (i.e., for any $y \in Y$) has **at least** one solution, we say that function f is **onto**, or **surjective**.

If such an equation always (i.e., for any $y \in Y$) has **at most** one solution, we say that function f is **one-to-one**, or **injective.**

If such an equation always (i.e., for any $y \in Y$) has **exactly** one solution, we say that function f is **bijective.** Sometimes such functions are also referred to as or **one-to-one correspondences**.

Bijections can be thought of as ways of identifying two different sets. In particular, if there exists a bijection f between two finite sets, $A \rightarrow B$, then $|A| = |B|$.

Inverse function

If f is a function from X to Y, $X \xrightarrow{f} Y$, then an inverse function for f, denoted by f^{-1}, is a function in the opposite direction, $Y \xrightarrow{f^{-1}} X$, with the property that a composition (a "round trip") returns each element to itself, $f^{-1}\big(f(x)\big) = f\big(f^{-1}(x)\big) = x$. Not every function has an inverse; those that do are called invertible. The inverse function exists if and only if f is a bijection.

In order to find an inverse function $g = f^{-1}$ we need to solve the equation $y = f(x)$ with respect to x,

$$x = g(y) \leftrightarrow solution\ of\ the\ equation\ y = f(x) \Rightarrow f\big(g(y)\big) = y \wedge g\big(f(x)\big) = x$$

<u>Example 1</u>. Let $f(x) = y = 3x + 2$. Solving with respect to x we obtain, $x = \frac{y-2}{3}$, so $g(y) = f^{-1}(y) = \frac{1}{3}y - \frac{2}{3}$, or, $f^{-1}(x) = \frac{1}{3}x - \frac{2}{3}$.

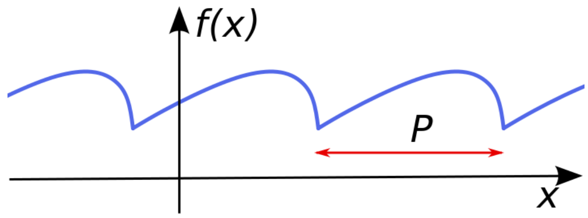
<u>Example 2</u>. If f converts a temperature in degrees Celsius C to degrees Fahrenheit F, the inverse function f^{-1} would be converting degrees Fahrenheit to degrees Celsius,

$$f(C) = \frac{9}{5}C + 32,\ f^{-1}(C) = \frac{5}{9}(F - 32).$$

<u>Exercise</u>. Suggest other examples of invertible functions and their inverses.

Periodic function

A function f on X is said to be **periodic** with period T if for some nonzero element $T \in X$ we have $f(x+T) = f(x)$ for all values of $\in X$. If there exists a least positive element T with this property, it is called the **prime period.** A function with period T will repeat on intervals of length T, and these intervals are sometimes also referred to as **periods.**



Geometrically, a periodic function can be defined as a function whose graph exhibits translational symmetry. A function f is periodic with period T if the graph of f is invariant under translation in the x-direction by a distance of T. A function that is not periodic is called **aperiodic.**

Monotonic function

A function f which is either entirely nonincreasing or nondecreasing is said to be **monotonic.**

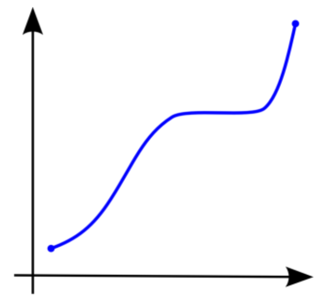


Figure 1.

Namely, a real-valued function f defined on a subset of the real numbers, $X \subseteq \mathbb{R}$, is called **monotonic** (also **monotonically increasing**, **increasing** or **non-decreasing**), if for all $x, y \in X$ such that $x \leq y$ one has $f(x) \leq f(y)$, so f preserves the order (see Figure 1). Likewise, a function is called **monotonically decreasing** (also **decreasing**, or **non-increasing**) if it reverses the order, i.e, for $x \leq y$, one has $f(x) \geq f(y)$, Figure 2.

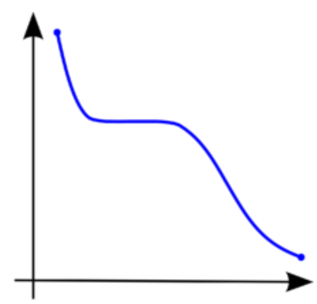


Figure 2.

If the order $\leq$ in the definition of monotonicity is replaced by the strict order $<$, then one obtains a stronger requirement. A function with this property is called **strictly increasing.** Again, by inverting the order symbol, one finds a corresponding concept called **strictly decreasing.** Functions that are strictly increasing or decreasing are

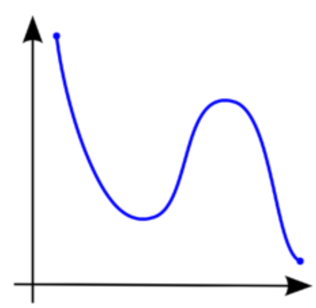


Figure 3.

one-to-one (because for x not equal to y, either $x < y$ or $x > y$ and so, by monotonicity, either $f(x) < f(y)$ or $f(x) > f(y)$, thus $f(x)$is not equal to $f(y)$).

Exercise. Give examples of monotonic functions.

Even and odd functions

Let $f(x)$ be a real-valued function of a real variable. Then f is **even** if the following equation holds for all x in the domain of f: $f(-x) = f(x)$.

Geometrically speaking, the graph of an even function is symmetric with respect to the Y-axis, meaning that it remains unchanged after reflection about the Y-axis.

$f(x)$ is **odd** if the following equation holds for all x in the domain of f: $f(-x) = -f(x)$, or $f(-x) + f(x) = 0$.

Geometrically, the graph of an odd function has rotational symmetry with respect to the origin, meaning that it remains unchanged after rotation of 180 degrees about the origin.

Exercise. Give examples of even and odd, periodic and aperiodic functions (cos(x), sin(x), …)?

Transformation of functions

- **Horizontal translation:**

$$g(x) = f(x + c)$$

The graph is translated c units to the left if $c > 0$ and c units to the right if $c < 0$.

- **Vertical translation:**

$$g(x) = f(x) + k$$

The graph is translated k units upwards if $k > 0$ and k units downwards if $k < 0$.

- **Change of scale:**

$$g(x) = f(ax)$$

The graph is "compressed" if $|a| > 1$ and "stretched out" if $|a| < 1$. In addition, if $|a| < 0$ the graph is reflected about the y-axis.

- **Change of amplitude:**

$$g(x) = Af(x)$$

The amplitude of the graph is increased by a factor of A if $|A| > 1$ and decreased by a factor of A if $|A| < 1$. In addition, if $|A| < 0$ the graph is inverted.

- **Reflection with respect to an axis:**

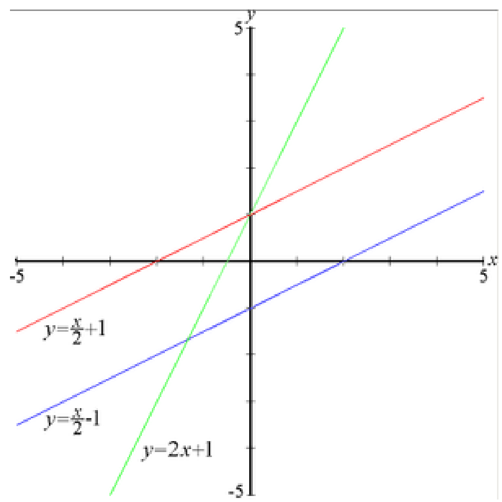
$g(x) = -f(x)$ reflects $f(x)$ over the X $-$axis

$g(x) = f(-x)$ reflects $f(x)$ over the Y $-$axis

Algebraic functions: linear function

The term linear function is used to mean a first-degree polynomial function of one variable. These functions are known as "linear" because their graphs in the Cartesian coordinate plane are straight lines. Such a function can be written as

$$f(x) = ax + b$$



called slope-intercept form, where a and b are real constants and x is a real variable. The constant a is often called the slope, while b is the Y-intercept, which gives the point of intersection between the graph of the function and the Y-axis. Changing a makes the line steeper or shallower, while changing b moves it up or down. Alternately, the linear function can be written as,

$$(y - y_1) = a(x - x_1)$$

$$Ax + By + C = 0$$

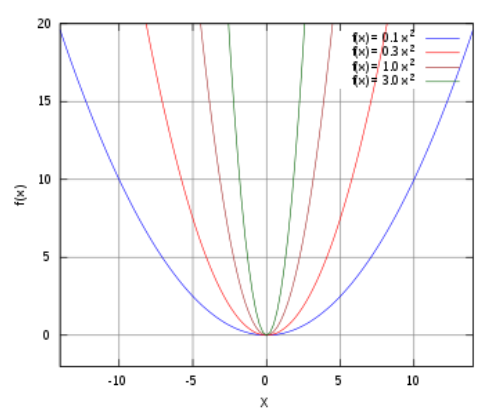
Examples. (a) $f(x) = \frac{x}{2} + 1$, (b) $f(x) = \frac{x}{2} - 1$, (c) $f(x) = 2x + 1$.

The graphs of these are shown in the figure.

Quadratic function

A **quadratic function** is a polynomial function of the form

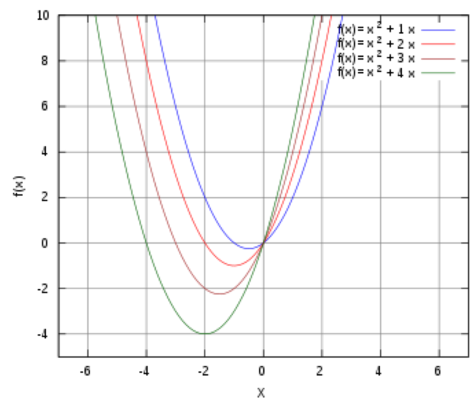
$$f(x) = ax^2 + bx + c, a \neq 0$$

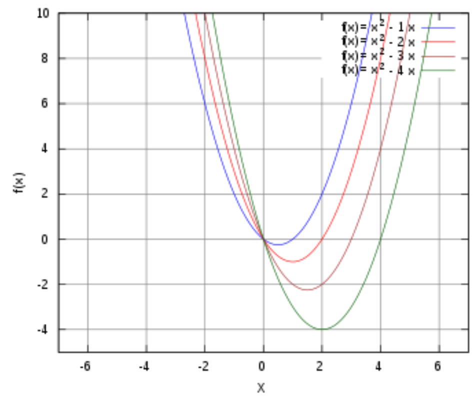


The graph of a quadratic function is a parabola whose axis of symmetry is parallel to the Y-axis.

The expression $ax^2 + bx + c$ in the definition of a quadratic function is a polynomial of degree 2 or second order degree polynomial, because the highest exponent of x is 2.

If the quadratic function is set equal to zero, then the result is a quadratic equation. The solutions to the equation are called the roots of the equation.





Forms of a quadratic function

A quadratic function can be expressed in three formats:

- $f(x) = ax^2 + bx + c$ is called the **standard form**,
- $f(x) = a(x - x_1)(x - x_2)$ is called the **factored form**, where x_1 and x_2 are the roots of the quadratic equation.
- $f(x) = a(x - h)^2 + k$ is called the **vertex form**, where h and k are the X- and Y-coordinates of the vertex, respectively.

Regardless of the format, the graph of a quadratic function is a parabola (as shown in the Figure above).

- If a is a positive number, $a > 0$, the parabola opens upward.
- If a is a negative number, $a < 0$ the parabola opens downward.

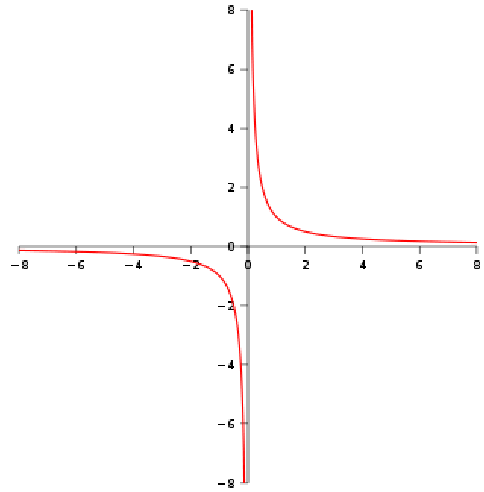
The coefficient a controls the speed of increase (or decrease) of the quadratic function from the vertex, bigger positive a makes the function increase faster and the graph appear more closed.

The coefficients b and a together control the axis of symmetry of the parabola (also the X-coordinate of the vertex) which is at $= -\frac{b}{2a}$.

The coefficient b alone is the declivity of the parabola as Y-axis intercepts.

The coefficient a controls the height of the parabola, more specifically, it is the point where the parabola intercept the Y-axis.

Rectangular hyperbola with horizontal/vertical asymptotes (Cartesian coordinates)



The rectangular hyperbola, $y = \frac{1}{x}$, is the inverse to linear function. A graph is shown in the figure.

Rectangular hyperbolas with the coordinate axes parallel to their asymptotes have the equation

$(x - h)(y - k) = m.$

The simplest example of rectangular hyperbolas occurs when the center (h, k) is at the origin: $y = \frac{m}{x}$

Powers and roots.

<u>General definition of powers</u>. For real $a \in R$ and natural $n \in N$,

$$a^n = a \cdot a \cdot \cdots \cdot a \ (n \ times).$$

This definition can be extended to all integers, $n \in Z$, by defining

$$a^0 = 1,$$

$$a^n = a^{-|n|} = \frac{1}{a^{|n|}} \ (n < 0).$$

Note that these two last statements are just adopted by definition; they cannot be proven, or established otherwise. By this definition,

$$a^{-n} = \frac{1}{a^n}, \forall \, n \in Z.$$

The convenience of the above definition is clear when we consider the law of addition (subtraction) and multiplication of powers,

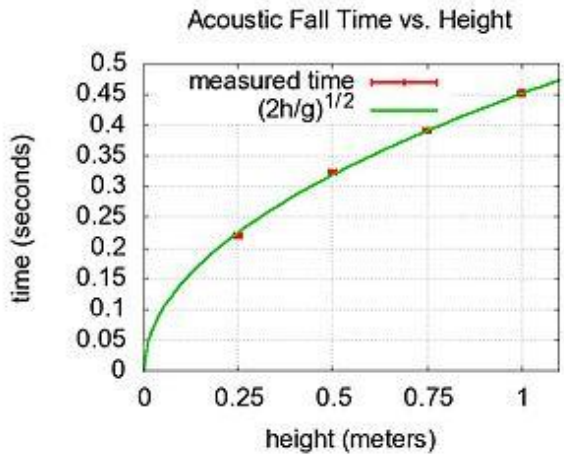
$$a^n \cdot a^m = a^{n+m}, \qquad \frac{a^n}{a^m} = a^n \cdot a^{-m} = a^{n-m},$$

$$(a^n)^m = a^{n \cdot m} = (a^m)^n \ \ (\forall \, n, m \in Z).$$

A **square root** of a number a, $y = \sqrt{a} = a^{1/2}$, is a number y such that $y^2 = a$.

The **arithmetic square root** of a number a is a positive number y such that $y^2 = a$.

Similarly, an **n-th degree root** of a number a, $y = \sqrt[n]{a} = a^{1/n}$, is a number y such that $y^n = a$. Thus the root-n function, $y = \sqrt[n]{x} = x^{1/n}$, is the inverse to the n-th degree power function, $y = x^n$.



For an odd n ($n = 2k + 1$) the power function $y = x^n$ is monotonously increasing for all x in the domain of definition ($x \in R$). Hence, it can be inverted within the full range, $y \in R$, which becomes the domain of definition of the root function. An odd-degree root function is defined both for positive and negative x, and is an odd function of x,

$$y(-x) = \sqrt[n]{-x} = -\sqrt[n]{x} = -y(x), \qquad x \in R, \qquad n = 2k + 1.$$

For an even n ($n = 2k$) the power function $y = x^n$ is monotonously decreasing for $x \le 0$ and increasing for $x \ge 0$ (its range consists of positive y only). Hence, it cannot be inverted within the full domain of definition ($x \in R$). For any y from its range ($y \ge 0$), the n-th degree parabola $y = x^n$ with even n has two values of x corresponding to it, one positive and one negative. Similarly to the case of a square root we can call the positive x satisfying $y = x^n$ the arithmetic n-th degree root of y, thus defining the n-th degree root function, whose domain of definition is $x \in R, x \ge 0$,

$$y(x) = \sqrt[n]{x} = x^{1/n}, \qquad x \in R^+ \ (x \ge 0), \qquad n \in N$$

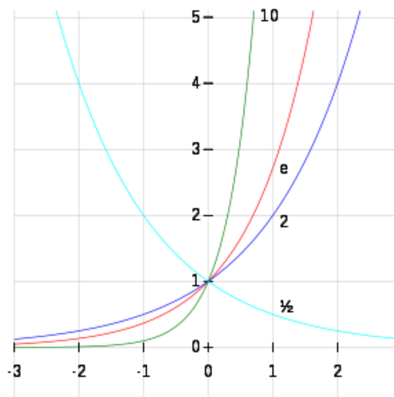
The following important properties of the arithmetic roots can be readily proven using the corresponding properties of powers:

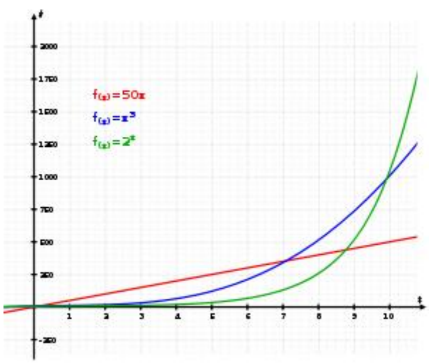
- $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
- $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ $(b \neq 0)$
- $\sqrt[n]{\sqrt[m]{a}} = \sqrt[n \cdot m]{a}$ $(m > 0)$
- $\sqrt[n]{a} = \sqrt[n \cdot m]{a^m}$ $(m > 0)$

- $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m \; (a \neq 0 \; if \; m \leq 0)$

Exponential function

Exponential functions are functions of the form $f(x) = a^x$ for a fixed base a, which could be any positive real number. Exponential functions are characterized by the fact that their rate of growth is proportional to their value. For example, suppose we start with a bank deposit m_0 such that its growth rate at any time is proportional to the amount of money in it. After t years the amount of money in the bank deposit will then be $m_0 a^t$ (an exponential function) where m_0 is the initial deposit and some $a > 1$ is the interest (growth) rate. Or, suppose we start with a population of cells such that its growth rate at any time is proportional to its size. The number of cells after t years will then be given by an exponential function a^t for some $a > 0$.



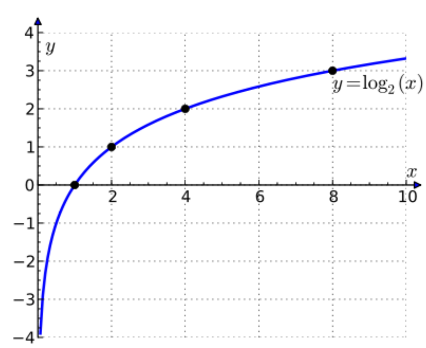


The graph illustrates how exponential growth (green) surpasses both linear (red) and cubic (blue) growth.

Exponential growth | Linear growth | Cubic growth

Exercise. What are the domain and range of exponential functions?

Logarithm



The **logarithm** of a real number is the exponent by which a given positive real number, the base, has to be raised to produce that number. For example, the logarithm of 1000 to base 10 is 3, because 1000 is 10 to the power 3: $1000 = 10^3$.

Definition. If $y = f(x) = b^x$, then $x = f^{-1}(y)$ is called the logarithm of y to base b, and is written $x = \log_b y$.

Examples. (a) $\log_{10} 1000 = 3$; (b) $\log_2 1024 = 8$; (c) $\log_3 27 = 3$.

Logarithms were introduced by John Napier in the early 17th century as a means to simplify calculations. They were rapidly adopted by scientists, engineers, and others to perform computations more easily and rapidly, using slide rules and logarithm tables. These devices rely on the fact - important in its own right - that the logarithm of a product is the sum of the logarithms of the factors.

Product, quotient, power, and root

The logarithm of a product is the sum of the logarithms of the numbers being multiplied; the logarithm of the ratio of two numbers is the difference of the logarithms. Therefore, the logarithm of the p-th power of a number is p times the logarithm of the number itself; the logarithm of a p-th root is the logarithm of the number divided by p. The following table lists these identities with examples:

	Formula	Example
product	$\log_b(xy) = \log_b(x) + \log_b(y)$	$\log_3(243) = \log_3(9 \cdot 27) = \log_3(9) + \log_3(27) = 2 +$
quotient	$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$	$\log_2(16) = \log_2\left(\frac{64}{4}\right) = \log_2(64) - \log_2(4) = 6 - 2$
power	$\log_b(x^p) = p\log_b(x)$	$\log_2(64) = \log_2(2^6) = 6\log_2(2) = 6$
root	$\log_b \sqrt[p]{x} = \frac{\log_b(x)}{p}$	$\log_{10}\sqrt{1000} = \frac{1}{2}\log_{10} 1000 = \frac{3}{2} = 1.5$

Change of base

The logarithm $\log_b x$ can be computed from the logarithms of x and b with respect to an arbitrary base k using the following formula:

$\log_b x = \frac{\log_k x}{\log_k b}$.