ASSIGNMENT 2: CONTINUED FRACTIONS

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Continued fractions

What is the best way to approximate an irrational number by fractions???

Consider the number $\pi \approx 3.1415926\dots$

As starting approximation, we can just take the whole part:

$$\pi = 3 + 0.1415\dots \approx 3$$

As next step, we now need to find a good approximation for 0.1415.... Let's invert it:

$$1/0.1415926\dots = 7.06251\dots \approx 7$$

so

$$\pi = 3 + \frac{1}{7.06251\dots} \approx 3 + \frac{1}{7} = \frac{22}{7}$$

This approximation has already been known to Archimedes.

If we want an even better approximation, we need to find a good rational approximation to $0.06251\dots$. Again, we use same trick:

$$1/0.06251 = 15.99\dots \approx 16$$

so $0.06251\dots \approx 1/16$, so we get

$$\pi = 3 + 0.1415926535\dots = 3 + \frac{1}{7.06251\dots} \approx 3 + \frac{1}{7 + \frac{1}{16}} = \frac{355}{113}$$

This is a very good approximation: the error is less that 0.0000003 (i.e. $3 * 10^{-7}$) - even though our fraction only has 3-digit denominator.

In general, we can apply this process to any irrational number. The approximations we will be getting in this way will have the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

Expressions of this form are called *continued fractions*. (To avoid huge formulas, which are also difficult to typeset, people commonly use notation $[a_0; a_1, a_2, a_3, \dots]$ for the continued fraction written above.)

It can be shown that rational approximations obtained in this way (with possible rounding of the last term) are "best" rational approximations in some precisely defined way.

Problems

1. Use the process above to get a continued fraction approximation for $\sqrt{2}$. Use it to get several good rational approximations to $\sqrt{2}$. [Hint: $\frac{1}{\sqrt{2}-1} = \sqrt{2}+1$.]

2. Can you guess what is the value of (infinite) continued fraction $[1; 1, 1, 1, 1, \dots]$, i.e.

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}$$

3. Do the same for $[2; 1, 2, 1, 2, \ldots]$, i.e.

$$2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \ldots}}}}$$

***4.** Show that any periodic continued fraction is a solution of a quadratic equation with integer coefficients.

[Hint: show that such a value x must satisfy an equation of the form

$$x = \frac{ax + b}{cx + d}$$

with integer a, b, c, d.

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