PIGEONHOLE PRINCIPLE

OCTOBER 17, 2021

Since many of you have been in the math club for a number of years, you might have seen some of these problems. If so, let me know, and I will give you other problems to work on.

The pigeonhole principle

If you put n items in m boxes, with $n > m$, then at least one box will have more than one item.

Generalization
If $n > km$ objects are put in m boxes, then at least one box will have more than k objects.

Problems

1. Given 5 points with integer coordinates in the plane, prove that one can always choose two of them so that the midpoint of the segment connecting them also has integer coordinates.

2. Consider the sequence of numbers 1, 11, 111, 1111, ...,
(a) Prove that among these numbers, there are two whose difference is divisible by 179
*(b) Prove that one of these numbers is divisible by 179.

3. Compute (by hand, using long division — it is important!) fractions $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$ as infinite decimals. Do you see any patterns?

4. Consider a sequence $a_1, a_2, a_3, \dots$ which is formed by the following rule: each next term a_{k+1} is obtained by multiplying a_k by 10 and then taking remainder upon division by 7. [Starting term a_1 is chosen arbitrarily.] Show that this will always produce a periodic sequence. What is the maximal period? What happens if instead of 7 we used another number, such as 11 or 12?

5. (a) Explain the relation between the two previous problems.
(b) Argue that any rational number p/q, when written in decimal, is periodic. What is the maximal period?

6. Let A be any set of 19 distinct integers chosen from the arithmetic progression $1, 4, 7, \dots, 100$. Prove that there must be two distinct integers in A whose sum is 104.

7. Five points are placed inside an equilateral triangle whose side has length one unit. Show that two of them may be chosen which are less than one half unit apart. What if the equilateral triangle is replaced by a square whose side has length of one unit?

8. Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint non-empty subsets whose members have the same sum.

[This problem is from 1972 International Math Olympiad, but it is one of the simplest IMO problems. As a hint, try first finding two different such subsets without requiring that they be disjoint.]

9. Given any $n + 1$ integers between 1 and $2n$, show that one of them is divisible by another. Is this best possible, i.e., is the conclusion still true for n integers between 1 and $2n$?