INEQUALITIES

OCTOBER 24, 2021

Inequalities

Today we will discuss some number of inequalities. The most famous of them is the arithmetic mean – geometric mean inequality:

Theorem. *If x, y are non-negative real numbers, then*

$$\frac{x+y}{2} \geq \sqrt{xy}$$

with equality if and only if $x = y$.

It can be used to find maximal value of xy if the sum $x + y$ is given, or minimal value of $x + y$ if the product xy is given.

It has a generalization:

Theorem. *If $x_1, \dots, x_n$ are non-negative real numbers, then*

$$\frac{x_1 + \dots + x_n}{n} \geq \sqrt[n]{x_1 \dots x_n}$$

with equality if and only if all x_i are equal.

Problems

1. (a) A rectangular garden is to be constructed using 20m of fence. What is the maximal area of such a garden?

(b) Same problem, but now the garden is adjacent to a wall, so we only need to use teh fence for the three remainign sides of the rectangle

2. What is the maximal area of a rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$? Sides of the rectangle should be parallel to coordinate axes.

Can you formulate and answer similar question for in 3 dimensions, for an ellipsoid?

3. Prove that for any $a, b, c > 0$, we have

$$(a+b)(b+c)(a+c) \geq 8abc$$

4. Prove that $n! < \left(\frac{n+1}{2}\right)^n$ for all $n > 1$.

5. let $g = \sqrt[n]{a_1 \dots a_n}$ be the geometric mean of positive numbers $a_1, \dots, a_n$. Prove that

$$(1 + a_1) \dots (1 + a_n) \geq (1 + g)^n$$

Hint: observe that when all a_i are equal, the inequality becomes an equality. Use that and AM-GM inequality to show that if we move away from this value by changing any pair (say a_1, a_2) while keeping the geometric mean unchanged, the left-hand side will increase.

6. Let us call a function $f(x)$ **convex** if, for any values a, b, the segment connecting points on the graph of f, namely $(a, f(a))$ and $(b, f(b))$, is above or coincides with the graph of f, but never goes below it.

For example, it is geometrically obvious (and can be proved algebraically) that the power function $f(x) = x^k$ is convex for non-negative x for $k \geq 1$, and also for $k < 0$ — in particular, function $1/x$ is convex for $x > 0$.

(a) Show that for a convex function,

$$\frac{f(a) + f(b)}{2} \geq f\left(\frac{a+b}{2}\right)$$

(b) Deduce the AM-GM inequality for 2 variables from the fact that the function $f(x) = x^2$ is convex.

(c) Prove that the maximal value of a convex function on any interval $[a, b]$ is achieved at one of endpoints.

(d) Apply the previous part to function $f(x) = 1/x$ to prove that for any $a, b, c \in [0, 1]$, one has

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} = \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1$$

Hint: if we fix a, b, for which value of c is this expression maximized?