INEQUALITIES 2

OCTOBER 31, 2021

Recall the Arithmetic Mean – Geometric Mean inequality.

Theorem. *If $x_1, \dots, x_n$ are non-negative real numbers, then*

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \dots x_n}$$

with equality if and only if all x_i are equal.

Today we also discussed another inequality, Cauchy inequality.

Theorem. *Let*

$$\mathbf{x} = (x_1, \dots, x_n)$$
$$\mathbf{y} = (y_1, \dots, y_n)$$

be two sequences of real numbers, of same length. Then

$$|\sum x_i y_i| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$$

where

$$\|\mathbf{x}\| = \sqrt{\sum x_i^2}$$

and similar for $\mathbf{y}$.

Proof of this inequality is given in Problem 4 below.

Problems

1. Warm-up exercise:
   (a) Prove that for any positive x,
   $$x + \tfrac{1}{x} \geq 2$$
   (b) Find the minimal value of $x + \frac{1}{2x}$

2. Let $a, b, c > 0$. Prove that then, $a^2 + b^2 + c^2 \geq ab + bc + ac$

3. Let $a, b, c > 0$. Prove that then,

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$$

4. Prove Cauchy inequality, using the following idea.
Show that the quantity

$$p(t) = \sum (x_i + t y_i)^2$$

is a quadratic polynomial in t and that the discriminant of this polynomial is ≤ 0. Deduce from this Cauchy inequality.

5. Find the maximal value of the function $2x + y + 3z$ on the unit sphere, i.e. the set of points (x, y, z) satisfying $x^2 + y^2 + z^2 = 1$.