MATH CLUB: INVARIANTS AND SEMIINVARIANTS

DEC 5, 2021

An invariant is something that does not change.
A semi-invariant is something that only changes in one direction (e.e.g, only decreases).

1. Numbers 1 through 20 are written on the blackboard. Every minute two of the numbers are erased and replaced by a single number: if the numbers were a, b, we replace them by $a+b$. Can you predict which number will be written on the board at the end?

2. Numbers 1 through 20 are written on the blackboard. Every minute two of the numbers are erased and replaced by a single number: if the numbers were a, b, we replace them by $a+b+ab$. Can you predict which number will be written on the board at the end?

3. 6 numbers are placed in the vertices of a regular hexagon. At each turn, you are allowed to add 1 to two numbers at adjacent vertices.

If initial collection of numbers was 1,0,1,0,0,0, can you make them all equal?

4. We place numbers +1 and -1 in the vertices of regular 12-gon. At every turn, you are allowed to change sign of numbers at 3 vertices in a row (e.g. vertices 5,6,7, or vertices 11, 12, 1). If initially one of the numbers was -1 and all others were 1, can you move -1 to the adjacent vertex by repeating this operation?

5. N non-negative real numbers are placed on a circle. Every minute, Sophia replaces each of these numbers a_i by absolute value of the difference $|a_i - a_{i+1}|$, where a_{i+1} is the next number going clockwise. [This is done simultaneously for all numbers on the circle.]

(a) Show that the quantity $A = \max(a_i)$ can only decrease with time.
(b) Show that after some number of repeating this operation, there will be at most two different values on the circle, 0 and some $a > 0$.
(c) Show that for $N = 4$, all numbers will stabilize to 0.
(d) Show that for $N = 3$, it is not necessarily so.
(e) Show that for any $N = 2^m$, all numbers will stabilize to 0.

6. In the country of RGB, there are 13 red, 15 green and 17 blue chameleons. Whenever two chameleons of different colors meet, both of them change their color to the 3rd one (e.g., if red and green meet, they both turn blue). Do you think it can happen that after some time, all chameleons become the same color?[Hint: give each color a numeric value, say 0, 1, 2]

7. We are given n red dots and n black dots on the plane, no three of them on the same line. Prove that then, one can connect them by n non-intersecting segments, each connecting a red dot with a black one. [Hint: consider all ways of connecting red and black dots in pairs; for each such connection, consider the total lenght of all segments. Show that if two segments intersect, then one can replace them by another pair to reduce the total length.]

8. King Arthur summoned $2n$ knights to his castle at Camelot. Each knight has at most $n-1$ enemies among the other knights. Prove that King Arthur can seat the knights around the round table so that no two enemies seat next to each other. [Hint: similar to the previous problem, try to find some quantity such that if two enemies seat next to each other, then one can reseat the knights to decrease this quantity. For example, you can take an arc of the circle and reverse the order of knights on this arc.]

9. A 100×100 yard field of wheat is divided into 1 yd $\times$ 1 yd squares. Initially, 9 of these squares were infected by some crop disease. The disease spreads as follows: for every square, if in the given year at least 2 of its 4 neighbors were infected, then next year the infection spreads to this square. (The squares that were infected stay infected forever). Prove that the disease will never spread to the whole field. [Hint: find semi-invariant!]

10. We have an infinite sheet of square ruled paper (think of it as first quadrant on the coordinate plane), with cells indexed by pairs of positive integers. In the beginning, we have a chip on square $(1,1)$. At every moment, we can make the following move: if there is a chip at square (i,j), and squares above and to the right of it (that is, squares $(i+1,j)$ and $(i,j+1)$) are both empty, we can remove the chip from (i,j) and put a chip in each of the squares $(i,j+1)$ and $(i+1,j)$.

Using these moves, can we clear the 3×3 square in the corner?

Hint to last problem: try to assign to each cell some non-negative real number ("weight") so that at each move, the total weight of all occupied cells is unchanged.