MATH CLUB
PROBABILITIES AND EXPECTED VALUES

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Expected values

If a quantity X depends on result of some random experiment (e.g. tossing a coin), taking values $a_1, a_2, \dots$ with probabilities $p_1, p_2, \dots$ respectively, the **expected value** of X is defined as

$$E(X) = p_1 a_1 + p_2 a_2 + \dots$$

Informally, if you repeat the experiment N times and denote by $X_1, X_2, \dots$ the values of X you received in these expereiments, then for large N, we have

$$\frac{X_1 + X_2 + \dots}{N} \approx E(X)$$

Important property of the expected value is that if you have two random variables X, Y (both depending on result of same random experiment), then

$$E(X + Y) = E(X) + E(Y)$$

regardless of whether X, Y are independent of each other or not. Note that it only works for the sum; for example, $E(XY)$ in general is not equal to $E(X)E(Y)$.

1. Three brothers want to toss a coin to decide which of them will be the first to play with a new toy.They only have a regular fair coin. How can they do it?

2. (a) You have a fair coin. How many times on average do you have to toss it to get heads? [Reformulation: consider experiment consisting of tossing a coin until you get heads. Find the expected value of the number of tosses.]
(b) Same, but for an unfair coin, which gives heads with probability p.

3. (a) How many times (on average) do you have to toss a fair coin until you get both heads and tails?
(b) How many times do you have to roll a die (regular six-sided fair die) until you get all 6 values?

4. How many times, on average, do you have to toss a fair coin before you see heads followed by tails?

5. n people, all of different heights, form a line. Since taller people block shorter people behind them, if you look at this line from the front, you will see fewer than n people. What is the expected number of people you will see?

Same problem appears in many other situations. E.g. if you have n cars on the road, where passing is not allowed, slower cars will block faster cars behind them, so you will have groups of cars traveling together, and you can ask to compute the average number of such groups.

6. A class has 13 boys and 17 girls. They form a line in random order. Then the teacher counts how many boys are standing behind girls (i.e., counts the number of girls immediately followed by a boy). What is the expected value of this number?

Hint: let X_i be the random variable which is equal to 1 if in the i-th position is a girl and in the $(i+1)$-st, a boy, and 0 otherwise.