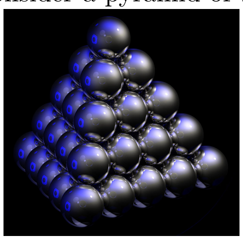
MATH CLUB: SEQUENCES AND FINITE DIFFERENCES

MARCH 6, 2022

1. Can you find a sequence a_n such that $a_n - a_{n-1} = n$? How many such sequences are there?
Can you do the same if instead we require that $a_n - a_{n-1} = n^2$?

2. Can you continue each of the following sequences?
(a) 3, 5, 7, 9,
(b) 1, 3, 6, 10, 15,
(c) -1, 2, 9, 22, 43, 74, 117
Can you also write a formula for n-th term for each of these sequences?

3. Consider a pyramid of balls, such as the one below:



How many balls does it contain if the number of layers is equal to n?

4. Let $f(x)$ be a function whose domain is the set of all integers (i.e., x is only allowed to be integer; values of $f(x)$ can be any numbers). For such a function, define the new function Δf by

$$(\Delta f)(x) = f(x) - f(x-1)$$

(function Δf is frequently called the "difference derivative" of f).
(a) Write explicit formula for $\Delta^2 f(x) = (\Delta(\Delta f))(x)$; for $\Delta^3 f$. Do you see the pattern? can you predict what $\Delta^5 f$ will be without computations? [Hint: Pascal Triangle.]
(b) Show that if $f(x)$ is polynomial of degree n, with leading coefficient a, then $(\Delta^n f)(x) = n!a$ is a constant, and $\Delta^{n+1} f(x) = 0$.
*(c) Show that conversely, if $\Delta^{n+1} f(x) = 0$, then f is a polynomial of degree at most n.

5. Each side of the triangle is divided into n equal parts. These points are connected by lines, parallel to the sides of the triangle (thus, we get 3 families of lines: parallel to side AB, parallel to side BC, parallel to side AC).
Into how many triangles do these lines divide the original triangle?

6. Each edge of a tetrahedron is divided into n equal parts. These points are connected by planes, parallel to the faces of the tetrahedron (thus, we get 4 families of planes — one family for each face of the tetrahedron).
Into how many tetrahedrons do these planes divide the original tetrahedron?

7. If we have n lines in a plane such that no no two lines are parallel, and no three lines intersect at a common point, into how many regions do these lines divide the plane?

8. A regular tetrahedron is rolled on a plane without slipping. Is it possible to roll it so that it comes back to the original postion in the plane, but resting on a different face?
[Hint: color the plane!]