# MATH CLUB: RECURRENT SEQUENCES 2

MARCH 20, 2022

Today, we discussed how one can write a general formula for a sequence defined by a recurrent relation — like the Fibonacci sequence, which is defined by

$$F_0 = 0, F_1 = F_2 = 1, \qquad F_{n+1} = F_n + F_{n-1} \tag{1}$$

**1.** Let us call a sequence $a_n$ a *generalized Fibonacci sequence* (GFC) if it satisfies the same recurrence relation ($a_{n+1} = a_n + a_{n-1}$), but might have different first two terms.

(a) Show that a geometric progression $1, r, r^2, \dots, a_n = r^n, \dots$, with $r \neq 0$, is a GFC if and only if $r$ satisfies the equation

$$r^2 = r + 1. \tag{2}$$

Find the roots of this equation.

(b) Let $r_1$, $r_2$ be the two roots of equation (2). Show that then any sequence of the form

$$a_n = c_1 r_1^n + c_2 r_2^n \tag{3}$$

(where $c_1, c_2$ are some constants that do not depend on $n$) is a GFC.

(c) Find constants $c_1, c_2$ so that the sequence $a_n$ defined by (3) satisfies $a_0 = 0$, $a_1 = 1$.

(d) Write a general formula for $F_n$.

**2.** Explain why for large $n$, $F_{n+1}/F_n$ is close to the *Golden Ratio*

$$\Phi = \frac{1 + \sqrt{5}}{2}$$

**3.** Pell numbers are defined by the relations

$$P_0 = 0, P_1 = 1, \qquad P_{n+1} = 2P_n + P_{n-1} \tag{4}$$

Try to modify the method of problem 1 to get a formula for Pell's numbers.

**4.** Show that for large $n$, the ratio $(P_{n-1} + P_n)/P_n$ is close to $\sqrt{2}$. Write the approximation one gets in this way for $n = 8$ and check how close it is to the actual value. (This series of approximations to $\sqrt{2}$ was known already in 4th century BC).