MATH CLUB: POLYNOMIALS

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Some useful facts about polynomials

- **Long division**: given polynomials $f(x)$, $g(x)$ (with degree of $g(x)$ at least 1), one can uniquely write $f(x)$ in the form
$$f(x) = q(x)g(x) + r(x), \quad \deg r(x) < \deg g(x)$$
  Polynomials $q(x)$, $r(x)$ are called *quotient* and *remainder* resepctively.
- **Bezout theorem**: when a polynomial $P(x)$ is divided by $(x-c)$, the remainder is $P(c)$. In particular, $P(x)$ is divisible by $(x-c)$ if and only if c is a root, i.e. $P(c) = 0$.
  Moreover, if $P(x)$ has integer coefficients and c is an integer root, then $P(x)$ is divisible by $(x-c)$ and the quotient has integer coefficients.

Vieta formulas

Suppose that we have a polynomial of degree n with leading coefficient 1 which has been completely factored:
$$p(x) = x^n + a_1x^{n-1} + \cdots + a_n = (x - x_1)\dots(x - x_n)$$
(thus, the roots of $p(x)$ are $x_1, \dots, x_n$).

Then one can express the coefficients $a_1, \dots, a_n$ in terms of roots $x_1, \dots, x_n$:
$$\begin{aligned}
a_1 &= -(x_1 + x_2 + \cdots + x_n),\\
a_2 &= x_1x_2 + \dots \qquad \text{(sum of products of all distinct pairs of roots)}\\
a_3 &= -x_1x_2x_3 + \dots \qquad \text{(sum of products of all distinct triples of roots)}\\
&\dots\\
a_n &= (-1)^n x_1 \dots x_n
\end{aligned}$$

These are called *Vieta formulas*. For $n = 2$, they become the usual formulas for quadratic equation: if $p(x) = x^2 + px + q = (x - x_1)(x - x_2)$, then $p = -(x_1 + x_2)$, $q = x_1x_2$.

Note that each of a_i is a symmetric expression in $x_1, \dots, x_n$: if one permutes $x_1, \dots, x_n$, the value of a_i doesn't change. In fact, it is also known that conversely, any symmetric polynomial of $x_1, \dots, x_n$ can be expressed in terms of a_i.

Problems

1. Find the remainder when $x^{13} + 1$ is divided by $x - 1$

2. The polynomial $P(x)$ has remainder 99 when divided by $x - 19$ and remainder 19 when divided by $x - 99$. What is the remainder when $P(x)$ is divided by $(x - 19)(x - 99)$?

***3.** Does there exist a polynomial with integer coefficients $P(x)$ such that for every integer n, $P(n)$ is a prime number?

4. Let x_1, x_2 be roots of the equation $x^2 + 13x - 7 = 0$. Find
(a) $x_1 + x_2$
(b) $\frac{1}{x_1} + \frac{1}{x_2}$
(c) $x_1^2 + x_2^2$
(d) $x_1^3 + x_2^3$

5. (2003 AMC 10A #18) What is the sum of the reciprocals of the roots of the equation
$$\tfrac{2003}{2004}x + 1 + \frac{1}{x} = 0$$

6. (2005 AMC 10B #16) The quadratic equation $x^2 + mx + n = 0$ has roots that are twice those of $x^2 + px + m = 0$, and none of m, n, p is zero. What is the value of n/p?

7. (2006 AMC 10B #14) Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + (1/b)$ and $b + (1/a)$ are the roots of the equation $x^2 - px + q = 0$. What is q?

8. (2001 AMC 12 #19) The polynomial $P(x) = x^3 + ax^2 + bx + c$ has the property that the mean of its zeros, the product of its zeros, and the sum of its coefficients are all equal. The y-intercept of the graph of $y = P(x)$ is 2. What is b?

9. One of the roots of equation $x^3 - 6x^2 + ax - 6 = 0$ is equal to 3. Find the other two roots.

10. Solve the system of equations:

$$\begin{aligned} x + y + z &= 6 \\ \tfrac{1}{x} + \tfrac{1}{y} + \tfrac{1}{z} &= \tfrac{11}{6} \\ xy + xz + yz &= 11 \end{aligned}$$

11. (1983 AIME) What is the product of the real roots of the equation

$$x^2 + 18x + 30 = \sqrt{x^2 + 18x + 45}$$

12. (1984 USAMO) The product of two of the four zeros of the quartic equation

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

is -32. Find k.