MATH CLUB: SYMMETRIC POLYNOMIALS

APRIL 2, 2022

Vieta formulas and symmetric polynomials

Suppose that we have a polynomial of degree n with leading coefficient 1 which has been completely factored:

$$p(x) = x^n + a_1 x^{n-1} + \dots + a_n = (x - x_1) \dots (x - x_n)$$

(thus, the roots of $p(x)$ are $x_1, \dots, x_n$).

Then one can express the coefficients $a_1, \dots, a_n$ in terms of roots $x_1, \dots, x_n$:

$$\begin{aligned}
a_1 &= -(x_1 + x_2 + \dots + x_n), \\
a_2 &= x_1 x_2 + \dots \qquad \text{(sum of products of all distinct pairs of roots)} \\
a_3 &= -x_1 x_2 x_3 + \dots \qquad \text{(sum of products of all distinct triples of roots)} \\
&\dots \\
a_n &= (-1)^n x_1 \dots x_n
\end{aligned}$$

These are called *Vieta formulas.* For $n = 2$, they become the usual formulas for quadratic equation: if $p(x) = x^2 + px + q = (x - x_1)(x - x_2)$, then $p = -(x_1 + x_2)$, $q = x_1 x_2$.

Note that each of a_i is a symmetric expression in $x_1, \dots, x_n$: if one permutes $x_1, \dots, x_n$ in any way, the value of a_i doesn't change. In fact, converse is also true:

Theorem. *Let $p(x_1, \dots, x_n)$ be a polynomial with complex coefficients which is unchanged under any permutation of variables $x_1, \dots, x_n$ (such polynomials are called* symmetric*).* Then p can be written as polynomial of $a_1, \dots, a_n$ defined above.

For example, for $n = 2$, if we take $p(x_1, x_2) = x_1^2 + x_2^2$, we can write

$$p = x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1 x_2 = a_1^2 - 2a_2$$

where $a_1 = -(x_1 + x_2)$, $a_2 = x_1 x_2$.

Problems

1. If $a + b + c + d = 2$ and $a^{-1} + b^{-1} + c^{-1} + d^{-1} = 2$, prove that

$$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} + \frac{1}{1-d} = 2.$$

2. x, y, z are integers such that $x + y + z = 0$. Prove that then, $2(x^4 + y^4 + z^4)$ is a square.

3. We are given three **positive** real numbers a, b, c such that $abc = 1$ and

$$a + b + c > \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

Prove that exactly one of three numbers a, b, c is greater than 1.

[Hint: look at polynomial $p(x) = (x - a)(x - b)(x - c)$. What can you say about $p(1)$?]

4. Consider the equation $(z + 1)^n = (z - 1)^n$. How many complex solutions does it have? What is the sum of squares of all the solutions?

5. Let s_2 be the following symmetric polynomial in $x_1, \dots, x_n$

$$s_2 = x_1^2 + x_2^2 + \dots + x_n^2$$

Express s_2 in terms of elementary symmetric polynomilas $a_1, \dots, a_n$.

6. Let s_3 be the following symmetric polynomial in $x_1, \dots, x_n$

$$s_3 = x_1^3 + x_2^3 + \cdots + x_n^3$$

Express s_3 in terms of elementary symmetric polynomilas $a_1, \dots, a_n$.

Can you try and get a general formula for $s_k = \sum x_i^k$? Or at least argue why this formula would be the same for all n?

7. Let

$$f(x) = (x - x_1)(x - x_2) = x^2 + px + q, \qquad p = -(x_1 + x_2), \quad q = x_1x_2$$

Consider the following polynomial in x_1, x_2: $D = (x_1 - x_2)^2$.

(a) Show that D is symmetric, i.e. it is unchanged when we permute x_1, x_2.
(b) Show that $f(x)$ has a double root if and only if $D = 0$.
(c) Express D as a polynomial of p, q.

This problem explains why D naturally appears in the quadratic formula.

***8.** This is analog of the previous problem for a cubic polynomial. Let $f(x)$ be a cubic polynomial; for simplicity, we only consider the case when the sum of roots is zero:

$$f(x) = (x-x_1)(x-x_2)(x-x_3) = x^3+px+q, \qquad x_1+x_2+x_3 = 0, \quad p = x_1x_2+x_1x_3+x_2x_3, \quad q = -x_1x_2x_3$$

Consider the following polynomial in x_1, x_2, x_3:

$$D = \Big((x_1 - x_2)(x_1 - x_3)(x_2 - x_3)\Big)^2$$

(a) Show that D is symmetric, i.e. it is unchanged when we permute x_1, x_2, x_3.
(b) Show that $f(x)$ has a multiple root if and only if $D = 0$.
(c) Express D as a polynomial of p, q.