MATH CLUB: COMBINATORICS

APRIL 24, 2024

Basic rules

The number of ways to choose k items out of n if the order in which they are chosen matters is

$$ {}_nP_k = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}. $$

The number of ways to choose k items out of n if the order in which they are chosen doesn't matter is

$$ {}_nC_k = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}. $$

For example, the number of sequences of length n consisting of k zeros and $n-k$ ones is ${}_nC_k$ (this is equivalent to choosing k positions where we put zeros).

1. A language of some ancient tribe consists of 6 vowels and 8 consonants, and in each word, vowels and consonants must alternate. How many 9-letter words can there be in this language?

2. How many ways there are for 15 students to take seats in a classroom with 30 chairs?

3. How many ways are there to place 8 rooks on the chessboard so that no two can attack each other?

4. How many ways are there to select 6 cards from the regular card deck (4 suits, 13 cards in each suit) so that among the selected cards, there is a card of each fo 4 suits.

5. How many "words" of length 12 can you form using just 2 letters, A and B, if each letter must appear 6 times? What if you are allowed to use 3 letters, A, B, C, each appearing 4 times? Can you get a general formula for number of words using 3 letters, appearing k_1, k_2, k_3 times respectively (thus, total length is $n = k_1 + k_2 + k_3$)?

6. You are given m white balls and n black balls ($m > n$). How many way are there to arrange them in a line so that no two black balls are next to each other?

7. Consider a triangle which is formed using the same rule as the Pascal triangle, but starts with different numbers:

$$\begin{array}{ccccccccc} & & & 1 & & -1 & & & \\ & & 1 & & 0 & & -1 & & \\ & 1 & & 1 & & -1 & & -1 & \\ 1 & & 2 & & 0 & & -2 & & -1 \end{array}$$

Can you suggest a formula for the entries in the n-th row?

8. An ant moves along the real line, starting at the origin and each time moving one unit either to the left or to the right. He takes $2n$ steps and ends up again at the origin

(a) Show that the number of such paths is equal to the constant term in the expression $(x + x^{-1})^{2n}$.

(b) Show that this number is equal to ${}_{2n}C_n$.

9. An ant moves in the plane, starting at the origin and each time moving one unit to the left or to the right or up or down. He takes $2n$ steps and ends up again at the origin.

(a) Show that the number of such paths is equal to the constant term in the expression $(x + x^{-1} + y + y^{-1})^{2n}$.

*(b) Prove that this number is equal to $({}_{2n}C_n)^2$. (Hint: rotate the plane 45 degrees. Then each ant's step moves him both horizontally and vertically.)

Stars and Bars

8. How many ways are there to put the chorus of 26 people in the stands in two rows (top and bottom)? There are no restrictions on how many people should be in the top row and how many, in the second.

9. How many solutions does the equation $x_1 + x_2 + x_3 = 2022$ have if x_1, x_2, x_3 must be non-negative integers? what if we require them to be positive integers?

10. How many different monomials in 3 variables x, y, z of total degree n are there? in 4 variables?

11. How many different monomials in 3 variables x, y, z of total degree n are there if we additionally require that each variable appears with positive degree (i.e. we look for monomials $x^a y^b z^c$, $a > 0$, $b > 0$, $c > 0$, $a + b + c = n$).

12. How many ways there are to put 15 chairs in 4 rooms if every room must have at least one chair? (Chairs are all identical, chairs inside the room are not ordered.)

13. How many ways there are to put 15 people in 4 rooms if every room must have at least one person? (People are all different, people inside the room are not ordered.)