CATALAN NUMBERS AND MORE

MAY 1, 2022

Catalan numbers

Consider the sequence of numbers defined by

$$c_0 = 1$$
$$c_1 = c_0c_0 = 1$$
$$c_2 = c_1c_0 + c_0c_1 = 2$$
$$\dots$$
$$c_{k+1} = c_0c_k + c_1c_{k-1} + \cdots + c_kc_0 = \sum_{i=0}^{k} c_ic_{k-i}$$

These numbers are called *Catalan numbers* and appear in many places.

1. Compute first 6 Catalan numbers, up to c_6

2. Consider expression

$$x_1 * x_2 * \cdots * x_{n+1}$$

where $*$ is some binary non-associative operation.

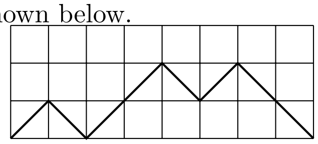
In order to make sense of this expression, we need to insert parentheses to indicate the order of operations. For example, for $n = 2$, there are two ways to do it: $(x_1 * x_2) * x_3$ and $x_1 * (x_2 * x_3)$.

(a) How many ways there are to put parentheses in product of 4 variables $x_1 * x_2 * x_3 * x_4$?

(b) Prove that there are exactly c_n ways to put parentheses in product of $n + 1$ variables [Hint: consider the operation performed last]

3. Prove that for a convex n-gon, there are exactly c_{n-2} ways to draw non-intersecting diagonals which would cut it into triangles. [Hint: choose an edge; look at the triangle containing this edge.]

4. A *Dyck path* is a polyline in the real plane which consists of segments $(1, -1)$ and $(1, 1)$ (i.e., moving diagonally: one unit to the right and one unit either up or down), starts at $(0, 0)$ and ends at $(2n, 0)$ and which never goes below the x-axis (but may touch it). An example of Dyck path with $n = 4$ is shown below.



We will denote the number of all Dyck paths of length $2n$ by D_n.

(a) Show that the number of Dyck paths of length $2n$ which are strictly above the x-axis (except the endpoints) is D_{n-1}.

(b) Show that $D_n = c_n$, i.e. the number of Dyck paths of length $2n$ is the Catalan number c_n (Hint: consider the first time the path touches the x-axis; use this point to divide the path into two subpaths).

(c) Show that the number of Dyck paths is the same as number of sequences of length $2n$, consisting of n symbols $+$ and n symbols $-$ such that in any initial segment of it, there are at least as many $+$ as $-$.

***5.** In this problem, you will prove that the number of Dyck paths of lenght $2n$ (and thus, the Catalan number c_n) is equal to

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

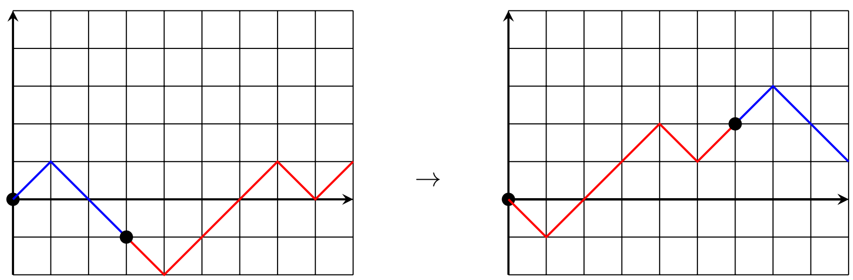
To do it, complete each of the steps below.

(a) Let S_n be the set of all paths consisting of n segments $(1, -1)$ (diagonally down) and $n + 1$ segments $(1, 1)$ (diagonally up), connecting points $(0, 0)$ and $(2n + 1, 1)$. Show that the number of such paths is $\binom{2n+1}{n}$.

(b) Let us call such a path *positive* if it is strictly above x-axis (except point $(0,0)$). Show that the number of positive paths is the same as the number of Dyck paths of length $2n$ and thus is equal to the Catalan number c_n.

(c) Consider the following operation on S_n, which we will call *rotation by* k: given an integer k, $0 \leq k \leq 2n+1$,
- given a path p, divide into two pieces p_1 (with $0 \leq x \leq k$) and p_2 (with $k \leq x \leq 2n+1$)
- translate p_2 so that it starts at $(0,0)$
- translate p_1 so that it starts at the endpoint of p_2

The picture below illustrates this operation (for $k=3$)



Note that for $k=0$ and $k=2n+1$, the rotation does nothing: it leaves the path unchanged. Prove that for every path in S_n, there is exactly one rotation that makes this path positive. [Hint: consider the lowest point on the path.]

(d) Let us group paths in S_n together if one can be obtained from another by some rotation. Prove that then each group has exactly $2n+1$ paths in it, and that in each group, there is exactly one positive path.

(e) Prove that the number of positive paths (and thus, the Catalan number c_n) is given by

$$\frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n}$$