MATH 10
ASSIGNMENT 8: VECTOR SPACES AND DIMENSION

NOVEMBER 21, 2021

Review of last time

Recall that we can systematically represent a system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

in terms of its augmented matrix

$$A|\mathbf{b} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & | & b_2 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & | & b_m \end{bmatrix}.$$

Using elementary row transformations, such an augmented matrix can be brought to the 'row echelon form", where each row begins with some number of zeroes, and each next row has more zeroes than the previous one:

$$\begin{bmatrix} X & * & * & * & * & * & * & | * \\ 0 & 0 & 0 & X & * & * & * & | * \\ 0 & 0 & 0 & 0 & X & * & * & | * \end{bmatrix}$$

(here X's stand for non-zero entries).

To solve such a system, we do the following:

- Variables corresponding to columns with X's in them are called pivot variables; the remaining ones are called free variables.
- Values for free variables can be chose arbitrarily. Values for pivot variables are then uniquely determined from the equations.

For example, in the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \\ x_2 + 3x_3 &= 6 \end{aligned} \tag{1}$$

varaibles x_1, x_2 are pivot, and variable x_3 is free, so we can solve it by letting $x_3 = t$, and then

$$\begin{aligned} x_2 &= 6 - 3x_3 = 6 - 3t \\ x_1 &= 5 - x_2 - x_3 = -1 + 2t \end{aligned}$$

Systems with no solutions

It could happen that a system of linear equations has no solutions. For example, if the augmented matrix is

$$\begin{bmatrix} 1 & 1 & | \ 2 \\ 0 & 0 & | \ 1 \end{bmatrix}$$

then the second equation reads $0 \cdot x_1 + 0 \cdot x_2 = 1$, which clearly has no solutions. It happens if in the row echelon form, there is an row which has all zero entries except the last one (corresponding to the right hand side of the equation), which is non-zero.

Lines and planes in $\mathbb{R}^3$

As we saw before, a single linear equation $ax_1+bx_2+cx_3 = d$ describes a plane perpendicular to the vector (a, b, c) in $\mathbb{R}^3$ (we assume that at least one of a, b, c is non-zero). If we use our system of linear equation techniques to solve this equation, we obtain something of the form

$$\mathbf{x} = \mathbf{a} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2,$$

where $t_1, t_2 \in \mathbb{R}$ are parameters which can take any real values and $\mathbf{a}, \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$ are some fixed vectors found by solving the equation. This is the *parametric* definition of a plane.

Likewise, the parametric definition of a line in three-dimensional space is

$$\mathbf{x} = \mathbf{a} + t\mathbf{v},$$

for some vectors $\mathbf{a}, \mathbf{v} \in \mathbb{R}^3$.

Vector Spaces

A *real vector space* is a set V together with two operations: vector addition

$$V \times V \to V : (\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} + \mathbf{w} \tag{2}$$

and multiplication by a scalar

$$\mathbb{R} \times V \to V : (a, \mathbf{v}) \mapsto a\mathbf{v}, \tag{3}$$

such that these operations satisfy the following properties hold:

$$\begin{aligned}
\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (\mathbf{u} + \mathbf{v}) + \mathbf{w} \\
\mathbf{u} + \mathbf{v} &= \mathbf{v} + \mathbf{u} \\
\exists \mathbf{0} \in V : \mathbf{0} + \mathbf{v} &= \mathbf{v}, \forall v \in V \\
\forall \mathbf{v} \in V \exists - \mathbf{v} \in V : \mathbf{v} + (-\mathbf{v}) &= \mathbf{0} \\
a(b\mathbf{v}) &= (ab)\mathbf{v} \\
1\mathbf{v} &= \mathbf{v}, \forall \mathbf{v} \\
a(\mathbf{u} + \mathbf{v}) &= a\mathbf{u} + a\mathbf{v} \\
(a + b)\mathbf{v} &= a\mathbf{v} + b\mathbf{v}.
\end{aligned}$$

Examples

I. The standard example is geometric vectors (arrows) in the plane or in Euclidean space. Addition is given by the parallelogram rule, and multiplication by a scalar by changing the length of the vector (and changing the direction if the number is negative).

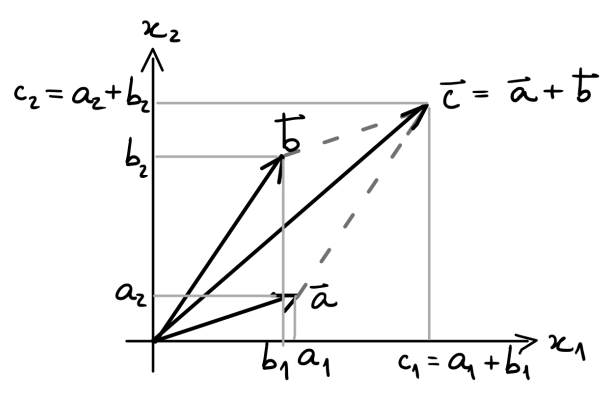

Figure 1. Vector addition in coordinates.

II. $\mathbb{R}^n$, the set of n-tuples of real numbers $x_1, x_2, \dots, x_n$. We will write them as a column of numbers:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

And the operations of addition of vectors and multiplication by numbers are

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

$$c \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}, \qquad c \in \mathbb{R}$$

III. $\mathbb{M}[m, n]$, the space of matrices of order $[m, n]$, is a generalization of the above concept: an array with m lines and n columns of the form:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

Sum of matrices of the same order and multiplication by a number are defined in analogy to the operations on $\mathbb{R}^n$ defined above.

IV. Polynomials in one variable of degree n (see problem).

By extending the ideas from the previous section, we will also refer to points of $\mathbb{R}^n$ as vectors (starting at the origin and ending at this point).

Basis Vectors

We commonly use coordinates to work with vectors. This is based on the fact that, given any two fixed noncolinear vectors $\vec{e}_1$, $\vec{e}_2$ (called *basis vectors*), we can write all vectors $\vec{a}$ in the plane as $\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2$ for some components (a_1, a_2). In terms of the components (called *coordinates*), we have

$$\vec{a} + \vec{b} = (a_1\vec{e}_1 + a_2\vec{e}_2) + (b_1\vec{e}_1 + b_2\vec{e}_2) = (a_1 + b_1)\vec{e}_1 + (a_2 + b_2)\vec{e}_2 \tag{4}$$

(see fig.1) and

$$c\vec{a} = c(a_1\vec{e}_1 + a_2\vec{e}_2) = (ca_1)\vec{e}_1 + (ca_2)\vec{e}_2. \tag{5}$$

Thus this gives a correspondence between vectors in the plane with addition and multiplication by a number as defined in **I**, and ordered pairs $(x_1, x_2) \in \mathbb{R}^2$ with the operations of addition and multiplication by numbers as defined in **II**. The concept that relates the two is that of *basis vectors.* The same goes for 3-dimensional vectors, which corresponds to $\mathbb{R}^3$.

Dimension

The number of vectors necessary and sufficient to form a basis is a property of the vector space, called the *dimension.* From the previous point we see that vectors in the plane form a *2-dimensional* vector space, and vectors in the euclidean space form a *3-dimensional* vector space. By using a basis, we can always identify a vector space of dimension d with $\mathbb{R}^d$ just as we did above.

The idea of dimension is very important in solving systems of linear equations. If a system of linear equations has solutions and has d free variables, it means that a general solution depends on the choice of d

numbers $t_1, \ldots, t_d$ — the values of free variables. Moreover, in this case the general solution can be written in the form

$$\mathbf{x} = \mathbf{a} + t_1\mathbf{v}_1 + \cdots + t_d\mathbf{v}_d$$

for some vectors $\mathbf{a}, \mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathbb{R}^n$. In this case we say that the set of solutions has *dimension* d. It can be shown that the dimension does not depend on how we brought the matrix to row echelon form.

For example, in the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \\ x_2 + 3x_3 &= 6 \end{aligned}$$

the general solution is

$$\mathbf{x} = \begin{bmatrix} -1+2t \\ 6-3t \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} + t\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

so the space of solutions has dimension 1.

Theorem. *If a system of linear equations has solutions, then the dimension of the space of solutions is given by*

$$d = (\textit{number of variables}) - (\textit{number of nonzero rows in row echelon form})$$

Indeed, the number of free variables is the number of all variables minus the number of pivot variables.

Thus, typically we expect that a system with n variables and k equations has $n-k$ dimensional space of solutions. This is not always true: it could happen that after bringing it to row echelon form, some rows become zero, or that the system has no solutions at all — but these situations are unusual (at least if $k \le n$).

Homework

1. Find a polynomial $p(x)$ of degree 3 which satisfies the following conditions

$$p(0) = 5, \quad p(1) = 2, \quad p(-1) = 4, \quad p(3) = -40.$$

2. Write the equation of a plane in $\mathbb{R}^3$ passing through the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. [Hint: if the equation of the plane is $ax_1 + bx_2 + cx_3 = d$, then plugging in it each of the points gives a condition on a, b, c, d; this gives a system of linear equations.]

Is such a plane unique?

3. Let P be the plane in $\mathbb{R}^3$ described parametrically as the set of all points of the form

$$\mathbf{x} = t_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad t_1, t_2 \in \mathbb{R}.$$

Write an equation of this plane. [Hint: find 3 points on this plane.]

4. Consider two planes $2x + 3y - z = 0$, $4x + z = 4$. Prove that their intersection is a line; write the line in the parametric form, by writing a generic point in the line as $\mathbf{x} = \mathbf{a} + t\mathbf{v}$ for some vector $\mathbf{v}$.

5. Find the intersection of 3 planes

$$\begin{aligned} x + 2y + 3z &= 3 \\ 3x + y + 2z &= 3 \\ 2x + 3y + z &= 3 \end{aligned}$$

6. Prove that the operations of addition and multiplication by a scalar in the vector space $\mathbb{R}^n$ (example **II**) satisfy all the properties outlined below equation (3).

7. The polynomials of degree n in one variable x form a vector space (look back at problem 1). Can you find a basis? What is the dimension?

8. Consider the space of matrices with 2 lines and 3 columns $\mathbb{M}[2,3]$. Can you find a basis for this space? What is the dimension? What about the general case $\mathbb{M}[m,n]$?