MATH 10
ASSIGNMENT 13: OPEN AND CLOSED SETS
JAN 9, 2022

Definition 1. A metric space is a set X with a distance function: for any $x, y \in X$ we have a real number $d(x, y)$ such that

1. $d(x, y) = d(y, x)$
2. $d(x, y) \geq 0$ for any x, y
3. $d(x, y) = 0$ if and only if $x = y$
4. Triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$.

Usual examples are $\mathbb{R}$, $\mathbb{R}^2$, ..., with the Euclidean metric $d(\vec{x}, \vec{y}) = \sqrt{(\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y})} = \sqrt{\sum (x_i - y_i)^2}$, but there are other examples as well (problem 1).

Given a point $x \in X$ and a positive real number ε, we define ε-neighborhood of x by

$$B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$$

(problem 2).

If $S \subset X$, denote by S' the complement of S. Then, for any $x \in X$, we can have one of three possibilities:

1. There is a neighborhood $B_\varepsilon(x)$ which is completely inside S (in paritcular, this implies that $x \in S$). Such points are called *interior points* of S; set of interior points is denoted by $\text{Int}(S)$.
2. There is a neighborhood $B_\varepsilon(x)$ which is completely inside S' (in particular, this implies that $x \in S'$). Thus, $x \in \text{Int}(S')$.
3. Any neighborhood of x contains points from S and points from S' (in this case, we could have $x \in S$ or $x \in S'$). Set of such points is called the *boundary* of S and denoted ∂S.

Note that in particular, the set S itself contains all points of $\text{Int}(S)$, some (possibly none) points of ∂S, and no points from $\text{Int}(S')$.

Definition 2. A set S is called *open* if every point $x \in S$ is an interior point: $S = \text{Int}(S)$.

A set S is called closed if $\partial S \subset S$.

Homework

1. Show that set $\mathbb{R}^2$ with distance defined by
$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$
is a metric space. (This distance is sometimes called *Manhattan* or taxicab distance — can you guess why?)

2. The Chebyshev distance in $\mathbb{R}^2$ is defined by
$$d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$
Draw the unit disc around the origin $B_1(0)$ using the Euclidean metric, then the Manhattan metric, then the Chebyshev metric in $\mathbb{R}^2$.

3. For each of the following subsets of $\mathbb{R}$, find its interior and boundary and determine if it is open, closed, or neither.
   (a) Set $\mathbb{N} = \{1, 2, 3, \dots\}$.
   (b) Interval $[0, 1]$
   (c) Open interval $(0, 1)$
   (d) Interval $[0, 1)$.
   (e) Set of all rational numbers
   (f) Set consisting of just two points $\{0, 1\}$
   *(g) Set $x^3 + 2x + 1 > 0$
   Are there any subsets of $\mathbb{R}$ which are both open and closed?

4. Show that a set S is open if and only if its complement S' is closed. [Hint: note that $\partial S = \partial S'$.]

***5.** For a set S, let $\overline{S} = S \cup \partial S = \{x \mid \text{ In any neighborhood of } x\text{, there are elements of } S\}$.

Prove that $\overline{S}$ is closed. (It is called the closure of S.)

6. Show that union and intersection of two open sets is open. Is it true if we replace two sets by any collection of open sets?

Same question about closed sets.