# MATH 10
## ASSIGNMENT 17: SERIES
FEB 27, 2022

### Series

Given a sequence $a_n$, consider a new sequence

$$\begin{aligned} &S_1 = a_1 \\ &S_2 = a_1 + a_2 \\ &S_3 = a_1 + a_2 + a_3 \\ &\dots \\ &S_n = a_1 + \dots + a_n = \sum_{i=1}^{n} a_i \end{aligned}$$

If the sequence $S_1, \dots, S_n$ has a limit, we will write

$$\sum_{i=1}^{\infty} a_i = \lim S_n$$

and call it the sum of the infinite series. In such a situation we say that the infinite series $\sum_1^\infty a_n$ converges.
For example:

$$1 + r + r^2 + \dots = \sum_{i=0}^{\infty} r^i = \frac{1}{1-r}, \qquad |r| < 1$$

Note that it is quite possible that the sequence $a_n$ converges but the series $\sum_1^\infty a_n$ does not converge!

### Problems

**1.** Prove that if the series $\sum_1^\infty a_n$ converges, i.e. the limit $\lim S_n$ exists, then $\lim a_n = 0$. [Hint: $a_n = S_n - S_{n-1}$ .]

**2.** Prove that if $0 \le a_n \le b_n$, then
(a) $\sum_{i=1}^n a_i \le \sum_{i=1}^n b_i$

(b) If the series $\sum_{i=1}^\infty b_i$ converges: $\sum_{i=1}^\infty b_i = B$, then the series $\sum_{i=1}^\infty a_i$ also converges, and $\sum_{i=1}^\infty a_i \le B$. [Hint: show that $S_n = \sum_{i=1}^n a_i$ is a bounded increasing sequence.]

**3.**
(a) Prove that the series $\sum \frac{1}{n(n+1)}$ converges and find the sum.
(b) Use the previous problem to prove that the series $\sum \frac{1}{n^2}$ converges.
[This problem is essentially a repetition of the last problem in the previous HW.]

**4.** Prove that the harmonic series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

does not converge. Hint: group the terms as follows:

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots + \frac{1}{8}\right) \dots$$

and show that the sum of terms inside each parentheses is $\ge 1/2$.

**5.** Prove that the series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

converges, by noticing that $\frac{1}{n!} \le \frac{1}{2^{n-1}}$.

The value of this series is denoted by letter $e$ and is at least as important in math as the number $\pi$:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.718281828\dots$$

(where we use the convention $0! = 1$)