

**MATH 10**  
**ASSIGNMENT 20: INTERMEDIATE VALUE THEOREM**  
MAR 27, 2022

**Definition.** A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is called continuous if, for every sequence  $a_n \in \mathbb{R}$  which has a limit  $\lim a_n = A \in \mathbb{R}$ , the sequence  $f(a_n)$  also has a limit and  $\lim f(a_n) = f(A)$ .

It was proved last time that the sum and product of continuous functions is continuous; the same is true for  $f/g$  as long as  $g \neq 0$ . In particular, all polynomials and rational functions are continuous everywhere where they are defined.

**Theorem** (Intermediate Value Theorem). *Let  $f(x)$  be a continuous function on the interval  $[a, b]$  such that  $f(a) < 0$  and  $f(b) > 0$ . Then there exists a point  $c \in (a, b)$  such that  $f(c) = 0$ .*

A proof was discussed in class.

HOMEWORK

1. Prove that polynomial  $x^3 + 3x - 2$  has a root between 0 and 5.
2. Prove that there exists a positive number  $x$  such that  $\sin(x) = 0.5x$ . (You can use without proof the fact that  $\sin(x)$  is continuous).
3. Let  $f(x) = x^{2n+1} + \dots$  be a polynomial of odd degree, with leading coefficient 1.
  - (a) Prove that for large enough  $x$ ,  $f(x) > 0$ . (i.e., there exists a real number  $M$  such that for all  $x \geq M$ ,  $f(x) > 0$ .)
  - (b) Prove that for large enough  $x$ ,  $f(-x) < 0$ .
  - (c) Prove that  $f(x)$  has at least one real root.
4. A traveler leaves town A at 9 am on Monday and arrives at town B at 4 pm the same day. He spends the night at town B, leaves it at 9 am on Tuesday, and returns to town A by 4 pm on Tuesday, following the same road.

Prove that there is a point on the road which he passed at exactly the same time on Monday and Tuesday.

Note that we are not assuming that the traveler goes at constant speed.
- \*5. Given a convex polygon  $S$  and a point  $A$  inside it, prove that there exists a chord of  $S$  which has  $A$  as the midpoint. [Hint: consider difference of lengths of the two pieces of a chord through  $A$  as a function of the angle.]
- \*6. We are given 10 red and 10 blue points in the plane, such that no three of them are on the same line. Prove that there is a line such that on each side of it there are 5 red and 5 blue points.