MATH 10
ASSIGNMENT 21: TOPOLOGICAL SPACES

APR 10, 2022

Many of the topics we studied, like open sets, continuous functions, and limits, can be generalized beyond the context of metric spaces. Today we look at the more general concept of a *topological space*.

Definition. A *topological space* is a set X with a collection $\mathcal{B}$ of subsets $N \subseteq X$, called neighborhoods, such that

1. Every point is in some neighborhood,

$$\forall x \in X, \exists N \in \mathcal{B} \text{ such that } x \in N$$

2. The intersection of any two neighborhoods of a point contains a neighborhood of the point,

$$\forall N_1, N_2 \in \mathcal{B} \text{ such that } x \in N_1 \cap N_2, \exists N_3 \in \mathcal{B} \text{ such that } x \in N_3 \subseteq N_1 \cap N_2$$

The set $\mathcal{B}$ is called a *basis* for the topology on the set X.

Definition. Let X be a topological space with basis $\mathcal{B}$. A subset $O \subseteq X$ is *open* if, for any $x \in O$, there exists a neighborhood $N \in \mathcal{B}$ of x contained in O. The set $\mathcal{T}$ of all open sets in X is called a *topology* on X.

A topological space is a set with a notion of open sets. It is all we need to define continuity:

Definition. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then a function $f : X \to Y$ is *continuous* if, for any $O_Y \in \mathcal{T}_Y$, $f^{-1}(O_Y) \in \mathcal{T}_X$. Moreover, if $f^{-1} : Y \to X$ exists and is also continuous, then f is said to be a *homeomorphism* and X and Y are then *homeomorphic*.

Note that a homeomorphism gives (1) a one-to-one correspondence between the elements of the sets and (2) a corresponding one-to-one correspondence between the open sets. In the homework problems, we will work with these definitions to understand how they relate to the case of metric spaces.

Homework

1. Let (X, d) be a metric space. Prove that the ϵ-neighborhoods $B_\epsilon(x)$ form a basis for the topology on X and that the definition of open sets with this basis coincides with our previous definition (in terms of interior points).

2. Given a set X, show that $\mathcal{B} = \{\{x\} | x \in X\}$ forms a basis for a topology in X. This is called the *discrete topology* in X. What are the open sets? Is there a metric d in X which leads to the same topology?

3. Let X be a topological space with topology $\mathcal{T}$ and basis $\mathcal{B}$. Then prove that
- (a) $\varnothing, X$ are open
- (b) The union of any collection of open sets is open
- (c) The intersection of any finite collection of open sets is open

This is usually taken as the definition of topological space, without reference to a basis.

4. Show that $\mathcal{T} = \{\varnothing, X\}$ satisfies the three properties of the previous problem. This is called the *trivial topology* in X. Is there a metric d in X which leads to the same topology?

5. Recall the limit definition of a continuous function $f : \mathbb{R} \to \mathbb{R}$: f is continuous if, for any sequence x_n which converges, the sequence $f(x_n)$ also converges and $\lim f(x_n) = f(\lim x_n)$. Is this equivalent to the defnition above? Explain (and ideally prove it).