MATH 5: HANDOUT 10
POWERS OF 2. BINARY NUMBERS.

Problem: If a certain population of bacteria doubles every day, and right now we have 1 gram of them, how much we will have in 2 days? in a week? in a month?

The answer: after 1 day we would have 2 grams; after 2 days, $2 \times 2 = 4$ grams; $\dots$; after n days, we will have $2 \times 2 \cdots \times 2$ (n times) grams. There is a **special notation** for this:

$$2^n = 2 \times 2 \cdots \times 2 \ (n \text{ times})$$

This grows very fast: for $n = 10$ (in ten days) we will have $2^{10} = 1,024$ grams; in another ten days, the amount will again multiply by 1,024, so we will have $1,024 \times 1,024 \approx 1,000,000$ grams, or one ton of bacteria; in 30 days, we will have about a thousand tons.

n	0	1	2	3	4	5	6	7	8	9	10
2^n	1	2	4	8	16	32	64	128	256	512	1024

In some problems, instead of multiplying by 2 every time, we are dividing by 2 every time:

Problem: guess a number between 1-100, asking questions that can only be answered "Yes" or "No".

Solution: the best strategy is doing it so that every question cuts the number of possibilities in half. So the first question should be "Is the number larger than 50?" If the answer is "Yes", we know that the number is between 51–100; if "No", it is between 1–50. Either way, there are now only 50 possible numbers. Next question should again cut the number of possibilities in half.

BINARY NUMBERS

Usual numbers are written in decimal notation, e.g., 351 means $3 \times 100 + 5 \times 10 + 1$

But we can also use powers of 2. For example, we can write a number 26 as $16 + 8 + 2 = 2^4 + 2^3 + 2^1 = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$. Note that the only digits we get are 0 and 1. Thus, we can encode this number by a sequence of digits 11010 (in binary!).

HOMEWORK

1. Solve the following equations:
   (a) $5(x-1) - 4 = 3x + 1$

   (b) $\frac{2}{3}(x-2) = -18$

   (c) $|2x - 1| = 7$

2. When Dennis was 27, his son was three years old. Now his sons's age is one third of Dennis' age. How is each of them now?

3. Find the sum $1 + 2 + 4 + \cdots + 2^n$ (the answer, of course, will depend on n). [Hint: first try computing it for several small values of n: find $1 + 2$, then $1 + 2 + 4$, then $1 + 2 + 4 + 8$. See if you can notice a pattern. After this, try formulating a general rule.]

4. Convert the decimal numbers to binary:
   9, 12, 24, 38, 45

5. Convert the following binary numbers to decimal:
   101, 1001, 10110, 11011, 10101

6. (a) How can one tell if a binary number is even or odd?
   (b) How can one tell if a binary number is divisible by 4?

7. There are 15 samples of water from various wells. It is known that exactly one of them contains a dangerous chemical. A lab can test water for the chemical, but the analysis is time-consuming and expensive. Can you find the sample containing the chemical using fewer than 15 tests? [Hint: you can take a drop of water from each of several samples and send the mix for analysis; then you would know if the chemical was in one of these samples.]