MATH 5: HANDOUT 23
GEOMETRY 4.

Special quadrilaterals: parallelogram, rhombus, trapezoid

Recall that a **parallelogram** is a quadrilateral in which opposite sides are parallel. Here are some of the properties of parallelograms; all of them were either proved before or will be proved in this homework.

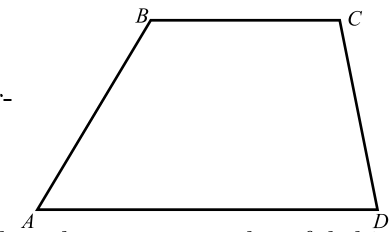
1. In a parallelogram, opposite sides are equal. Conversely, if $ABCD$ is a quadrilateral in which opposite sides are equal: $AB = CD$, $BC = AD$, then $ABCD$ is a parallelogram.

2. In a parallelogram, diagonals bisect each other, i.e. the intersection point of two diagonals is the midpoint of each of them. Conversely, if $ABCD$ is a quadrilateral in which diagonals bisect each other, then $ABCD$ is a parallelogram.

3. In a parallelogram, opposite angles are equal.

A **rhombus** is a quadrilateral in which all four sides are equal. By property 1 above, any rhombus is also a parallelogram, so all the above properties hold. In addition, in a rhombus diagonals are perpendicular (shown in previous homework).

A **trapezoid** is a quadrilateral in which one pair of opposite sides are parallel: $AD \parallel BC$. These parallel sides are usually called **bases**.



A trapezoid does not have as many useful properties as a parallelogram, but there are several useful things you will see in problem 7.

Warning: in this homework, more than ever, you will need results of previous exercises when doing the next one. So when doing, say, exercise 2, see if you can make use of exercise 1.

1. Let $ABCD$ be a quadrilateral such tath $AB = CD$, $AB \parallel CD$. Prove that then $ABCD$ is a parallelogram. [Hint: show that tirangles $\triangle ABD$, $\triangle CDB$ are congruent.]

2. Let $ABCD$ be a parallelogram, and let M, N be midpoints of sides AB, CD. Prove that then $AMND$ is a parallelogram, and deduce from this that $MN \parallel AD$, $MN = AD$.

3. (a) Prove that if in a quadrilateral $ABCD$ diagonals bisect each other (i.e., intersection point is the midpoint of each of the diagonals), then $ABCD$ is a parallelogram. [Hint: find some congruent triangles in the figure.]

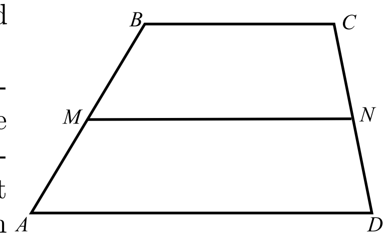
(b) Prove that if in a quadrilateral $ABCD$ diagonals bisect each other and are perpendicular, then it is a rhombus.

4. To check whether a piece of paper is a square, John folds it along a diagonal. If the corners match, he decides it is a square. Is he right? What if he folds along both diagonals?

5. Can you cut a trapezoid into pieces from which you can construct a rectangle?

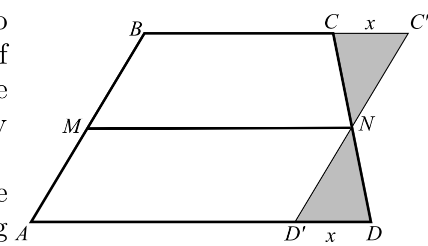
6. Suppose you have a large supply of tiles, all of the same size and shape — namely, a parallelogram. Can you tile a plane with these tiles? Can you find different ways of doing this? What if instead of a parallelograms you have trapezoids?

7. Let $ABCD$ be a trapezoid with bases AD, BC. Let M be midpoint of side AB and N — midpoint of side CD. Prove that $MN \parallel AD$ and $MN = \frac{AD+BC}{2}$.



Read the solution to this problem below and try your best to understand it. When you think that you understood everything, take a piece of paper and write a solution yourself without looking at the one written here. If you have questions about this problem that you can not answer yourself, bring them to the class. We will discuss this problem during the class as well.

Solution



Draw a line $C'D'$ through point N parallel to AB. Then the two shaded triangles are congruent by ASA, so N is also the midpoint of $C'D'$. On the other hand, $ABC'D'$ is a parallelogram, so MN is the line connecting midpoints of two sides of a parallelogram. Thus, by Problem 2 in this homework, $MN \parallel AD$, $MN = AD' = BC'$.

Denote $x = CC' = BB'$. Then $BC' = BC + x$, $AD' = AD - x$. Since $ABC'D'$ is a parallelogram, $BC' = AD'$, so $BC + x = AD - x$. Solving for x, we get $x = \frac{AD-BC}{2}$, so $MN = BC' = BC + x = \frac{AD+BC}{2}$.