MATH 5: HANDOUT 22
GEOMETRY 2.

Sum of angles of an n-gon

Recall that sum of angles of a triangle is $180°$. Since a quadrilateral can be cut into 2 triangles, sum of angles of a quadrilateral is $2 \times 180° = 360°$. Similarly, for a pentagon we get $3 \times 180°$, and for an n-gon, the sum of angles is $(n-2) \times 180°$.

Congruence

In general, two figures are called congruent if the have same shape and size. We use symbol $\cong$ for denoting congruent figures: to say that M_1 is congruent to M_2, we write $M_1 \cong M_2$.

Precise definition of what "same shape and size" means depends on the figure. Most importantly, for triangles it means that the corresponding sides are equal and corresponding angles are equal: $\triangle ABC \cong \triangle A'B'C'$ is the same as:
$AB = A'B'$, $BC = B'C'$, $AC = A'C'$,
$\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.

Note that for triangles, the notation $\triangle ABC \cong \triangle A'B'C'$ not only tells that these two triangles are congruent, but also shows which vertex of the first triangle corresponds to which vertex of the second one. For example, $\triangle ABC \cong \triangle PQR$ is not the same as $\triangle ABC \cong \triangle QPR$.

Congruence tests for triangles

By definition, to check that two two triangles are congruent, we need to check that corresponding angles are equal and corresponding sides are equal; thus, we need to check 6 equalities. However, it turns out that in fact, we can do with fewer checks.

Axiom 1 (Side-Side-Side rule). *If $AB = A'B'$, $BC = B'C'$ and $AC = A'C'$ then $\triangle ABC \cong \triangle A'B'C'$.*

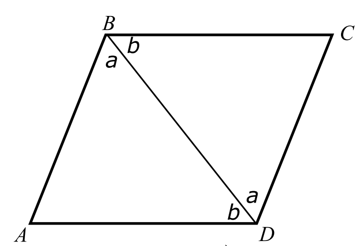
This rule is commonly referred to as SSS rule.

One can also try other ways to define a triangle by three pieces of information, such as two sides and an angle between them. We will discuss it next time.

This rule — and congruent triangles in general — are very useful for proving various properties of geometric figures. As an illustration, we prove the following useful result.

Theorem. *Let $ABCD$ be a quadrilateral in which opposite sides are equal: $AB = CD$, $AD = BC$. Then $ABCD$ is a parallelogram.*

Proof. Let us draw diagonal BD. Then triangles $\triangle ABD$ and $\triangle CDB$ are congruent by SSS; thus, two angles the two angles labeled by letter a in the figure are equal; also, two angles labeled by letter b are also equal. Thus, lines BC and AD are parallel (alternate interior angles!). In the same way we can show that lines AB and CD are parallel. Thus, $ABCD$ is a parallelogram. □



(It is also true in the opposite direction: in a parallelogram, opposite sides are equal.)

Homework

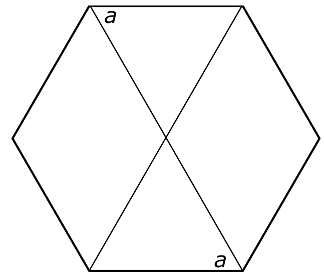
1. Let CD be a continuation of side AC in a triangle $\triangle ABC$. Show that then $\angle BCD = \angle A + \angle B$ (such an angle is sometimes called an *exterior angle* of the triangle. [Hint: sum of the angles in a triangle is equal to 180°.]

2. An n-gon is called *regular* if all sides are equal and all angles are also equal.

(a) How large is each angle in a regular hexagon (6-gon)?

(b) Show that in a regular hexagon, opposite sides are parallel. (This is the reason why this shape is used for nuts and bolts).

[Hint: show that each of the angles labeled by letter a in the figure is equal to 60°, and then use theorem about alternate interior angles.]

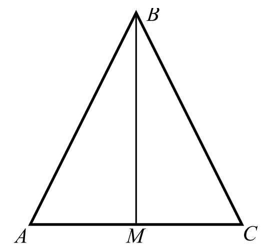


3. Let ABC be a triangle in which two sides are equal: $AB = BC$ (such a triangle is called *isosceles*). Let M be the midpoint of the side AC, i.e. $AM = MC$.

(a) Show that triangles $\triangle ABM$ and $\triangle CBM$ are congruent.

(b) Show that angles $\angle A$ and $\angle B$ are equal

(c) Show that $\angle AMB = 90°$ (hint: $\angle AMB = \angle CMB$).



4. Let $ABCD$ be a quadrilateral such that $AB = BC = CD = AD$ (such a quadilateral is called rhombus). Let M be the intersection point of AC and BD.

(a) Show that $\triangle ABC \cong \triangle ADC$

(b) Show that $\triangle AMB \cong \triangle AMD$

(c) Show that the diagonals are perpendicular and that the point M is the midpoint of each of the diagonals.

[Hint: after doing each part, mark on the figure all the information you have found — which angles are equal, which line segments are equal, etc: you may need this information for the following parts.]