

MATH 6: HANDOUT 20
GEOMETRY IV

1. TRIANGLE INEQUALITIES

In this section, we use previous results about triangles to prove two important inequalities which hold for any triangle.

We already know that if two sides of a triangle are equal, then the angles opposite to these sides are also equal (Theorem 12). The next theorem extends this result: in a triangle, if one angle is bigger than another, the side opposite the bigger angle must be longer than the one opposite the smaller angle.

Theorem 15. *In $\triangle ABC$, if $m\angle A > m\angle C$, then we must have $BC > AB$.*

Proof. Assume not. Then either $BC = AB$ or $BC < AB$.

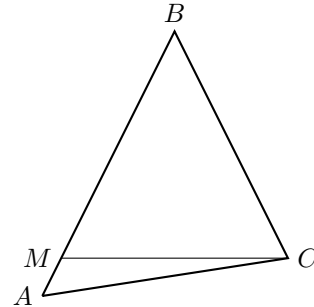
But if $BC = AB$, then $\triangle ABC$ is isosceles, so by Theorem 13, $m\angle A = m\angle C$ as base angles, which gives a contradiction.

Now assume $BC < AB$, find the point M on AB so that $BM = BC$, and draw the line MC . Then $\triangle MBC$ is isosceles, with apex at B . Hence $m\angle BMC = m\angle MCB$. On the other hand, we have $m\angle BMC > m\angle A$, and by Axiom 3, we have $m\angle C = m\angle ACM + m\angle MCB > m\angle MCB$, so

$$m\angle C > m\angle MCB = m\angle BMC > m\angle A$$

so we have reached a contradiction.

Thus, assumptions $BC = AB$ or $BC < AB$ both lead to a contradiction. Therefore, the only possibility is that $BC > AB$. □



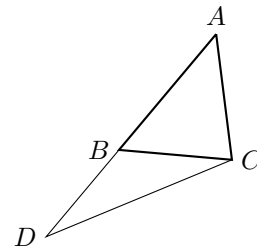
The converse of the previous theorem is also true: opposite a long side, there must be a big angle.

Theorem 16. *In $\triangle ABC$, if $BC > AB$, then we must have $m\angle A > m\angle C$.*

The following theorem doesn't quite say that a straight line is the shortest distance between two points, but it says something along these lines. This result is used throughout much of mathematics, and is referred to as "the triangle inequality".

Theorem 17 (The triangle inequality). *In $\triangle ABC$, we have $AB + BC > AC$.*

Proof. Extend the line AB past B to the point D so that $BD = BC$, and join the points C and D with a line so as to form the triangle ADC . Observe that $\triangle BCD$ is isosceles, with apex at B ; hence $m\angle BDC = m\angle BCD$. It is immediate that $m\angle DCB < m\angle DCA$. Looking at $\triangle ADC$, it follows that $m\angle D < m\angle C$; by Theorem 15, this implies $AD > AC$. Our result now follows from $AD = AB + BD$ (Axiom 2) □



2. SPECIAL QUADRILATERALS

Among all quadrilaterals, there are some that have special properties. In this section, we discuss three such types.

Definition 1. A quadrilateral is called

- a parallelogram, if both pairs of opposite sides are parallel
- a rhombus, if all four sides have the same length
- a trapezoid, if one pair of opposite sides are parallel (these sides are called bases) and the other pair is not.

These quadrilaterals have a number of useful properties.

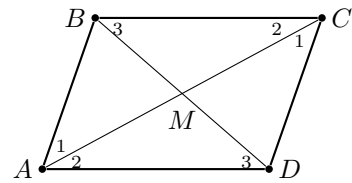
Theorem 18. *Let $ABCD$ be a rhombus. Then the diagonals are perpendicular, and the intersection point of diagonals is the midpoint for each of them.*

The proof of this theorem was one of the problems in your previous homework assignment.

Theorem 19. *Let $ABCD$ be a parallelogram. Then*

- $AB = CD$, $AD = BC$
- $m\angle A = m\angle C$, $m\angle B = m\angle D$
- The intersection point M of diagonals AC and BD bisects each of them.

Proof. Consider triangles $\triangle ABC$ and $\triangle CDA$ (pay attention to the order of vertices!). By alternate interior angles theorem, angles $\angle CAB$ and $\angle ACD$ are equal (they are marked by 1 in the figure); similarly, angles $\angle BCA$ and $\angle DAC$ are equal (they are marked by 2 in the figure). Thus, by ASA, $\triangle ACB \cong \triangle CAD$. Therefore, $AB = CD$, $AD = CB$, $m\angle B = m\angle D$, and $m\angle A = m\angle C$.

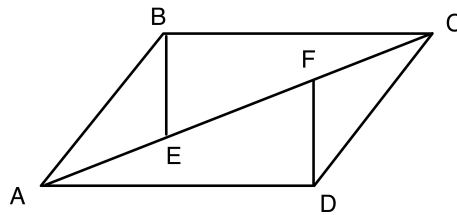
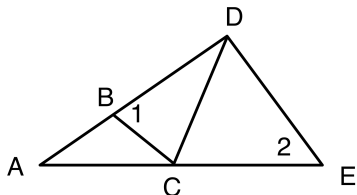


Now let us consider triangles $\triangle AMD$ and $\triangle CMB$. In these triangles, angles labeled 2 are congruent (discussed above), and by alternate interior angles theorem, angles marked by 3 are also congruent; finally, $AD = BC$ by previous part. Therefore, $\triangle AMD \cong \triangle CMB$ by ASA, so $AM = MC$, $CM = MD$. \square

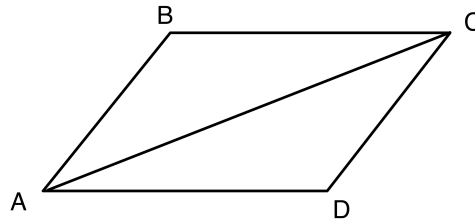
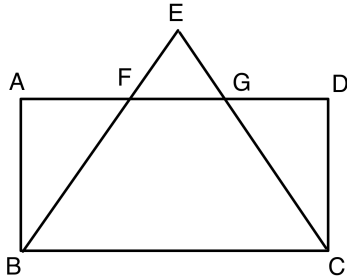
Theorem 20. Let $ABCD$ be a quadrilateral such that opposite sides are equal: $AB = CD$, $AC = BD$. Then $ABCD$ is a parallelogram.

HOMework

- Given that $\angle 1 \cong \angle 2$, prove that $AD > ED$. Hint: Find how does $m\angle DAE$ compares to $m\angle 1$, and then use Theorem 15.
- In the following parallelogram we have that $AE = CF$, prove that $\angle ABE \cong \angle CDF$.

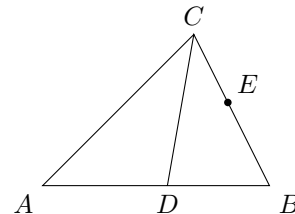


- Given the rectangle $ABCD$, and knowing that $BE = CE$, prove that $AF = DG$.
- In the following figure $ABCD$ is a parallelogram in which $AD > DC$. Prove that $m\angle BAC > m\angle DAC$.



- Let $ABCD$ be a rectangle (i.e., all angles have measure 90°). Show that then, opposite sides are equal.
- Prove Theorem 20
- Show that each rhombus is a parallelogram.

- Given $\triangle ABC$, let D be on AB such that $\angle ACD$ is equal to $\angle BCD$. Suppose we wish to place a point E on \overline{BC} such that $\triangle CED$ is isosceles. Prove then that we must have $AC \parallel DE$.



- In a trapezoid $ABCD$, with bases AD , BC , let E be the midpoint of side AB and F be the midpoint of side CD . Show that $EF \parallel AD$. [Optional: Can you prove that $EF = \frac{1}{2}(AB + CD)$? The figure on the right may help.]

