MATH 6: HANDOUT 12
GEOMETRIC SEQUENCES

A sequence of numbers is a **geometric sequence** or **geometric progression** if if the next number in the sequence is the current number times a fixed constant called the **common ratio** or q.

Example: The sequence $6, 12, 24, 48, \ldots$ is a geometric sequence because the next number is obtained from the previous by multiplication by $q = 2$.

We can also find the n-th term if we know the 1st term and q.

Example: What is a_{10} in the example above?

$$\begin{aligned} a_1 &= 6 \\ a_2 &= a_1 q = 6 \cdot 2 = 12 \\ a_3 &= a_2 q = (a_1 q)q = a_1 q^2 = 6 \cdot 2^2 = 24 \end{aligned}$$

The pattern is:

$$\begin{aligned} a_n &= a_1 q^{n-1} \\ a_{10} &= a_1 q^9 = 6 \cdot 2^9 = 6 \cdot 512 = 3072 \end{aligned}$$

Properties of a Geometric Sequence. Any term is the **geometric mean** of its neighbors:

$$a_n = \sqrt{a_{n-1} \cdot a_{n+1}}$$

Proof:

$$\begin{aligned} a_n &= a_{n-1} q \\ a_n &= a_{n+1}/q \end{aligned}$$

Multipluying these two equalities gives us:

$$a_n^2 = a_{n-1} \cdot a_{n+1}$$

from where we can get what we need.

Sum of a Geometric Sequence. Let's try to sum $1 + 2 + 4 + \cdots + 64$. For purposes of working with this sum, let it be called S, i.e. $S = 1 + 2 + 4 + \cdots + 64$. Then I can notice that $2S = 2 + 4 + 8 + \cdots + 128$; subtract the original sum to get $2S - S = 128 - 1$ (everything else cancels out). Thus $S = 127$. What did we do here? We multiplied by 2, which lined up the terms of the sequence to the next term over. In the geometric sequence 1, 2, ..., 64, the common ratio is $q = 2$.

Let's do this in general. Let $a_1, \ldots, a_n$ be a geometric sequence with common ratio q.

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \frac{a_1(1 - q^n)}{1 - q}$$

Proof: To prove this, we write the sum and multiply it by q:

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_n \\ qS_n &= qa_1 + qa_2 + \cdots + qa_n \end{aligned}$$

Now notice that $qa_1 = a_2, \ldots qa_2 = a_3, \ldots, qa_n = a_{n+1}$, etc, so we have:

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_n \\ qS_n &= a_2 + a_3 + \cdots + a_{n+1} \end{aligned}$$

Subtracting the second equality from the first, and canceling out the terms, we get:

$$\begin{aligned} S_n - qS_n &= (a_1 - a_{n+1}), \text{ or} \\ S_n(1 - q) &= (a_1 - a_1 q^n) \\ S_n(1 - q) &= a_1(1 - q^n) \end{aligned}$$

from which we get the formula above.

Homework problems

1. Write out the first four terms of each of the following geometric sequences, given the first term b_1 and common ration q.

(a) $b_1 = 1$ and $q = 3$
(b) $b_1 = 1$ and $q = \frac{1}{2}$
(c) $b_1 = -10$ and $q = \frac{1}{2}$
(d) $b_1 = 27$ and $q = -\frac{1}{3}$

2. What are the first two terms of the geometric progressions $a_1, a_2, 24, 36, 54, \ldots$?

3. Find the common ratio of the geometric progressions $1/2, -1/2, 1/2, \ldots$. What is a_{100}?

4. Calculate:

$$S = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^{10}}.$$

5. Calculate:

$$S = 1 - 2 + 2^2 - 2^3 + 2^4 - 2^5 + \cdots - 2^{15}$$

6. Calculate

$$S = 1 + 3 + 9 + 27 + 81 + 243,$$

first via the method of multiplying by the common ratio, then by plugging into the formula directly. Which method do you like better?

7. Calculate

$$S = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243}.$$

8. A geometric progression has 99 terms, the first term is 12 and the last term is 48. What is the 50th term?

9. Simplify the following expression

$$1 + x + x^2 + x^3 + \ldots + x^{100}$$

10. If we put one grain of wheat on the first square of a chessboard, two on the second, then four, eight, ..., approximately how many grains of wheat will there be? (you can use an approximation $2^{10} = 1024 \approx 10^3$).

Can you estimate the total volume of all this wheat and compare with the annual wheat harvest of the US, which is about 2 billion bushels. (A grain of wheat is about 10 mm^3; a bushel is about 35 liters, or 0.035 m^3)