MATH 7: HOMEWORK 12

General quadratic formula

January 16, 2022

1. Quadratic equation in a standard form.

Standard form: $ax^2 + bx + c = 0$

A **quadratic equation** could have

- no solution,
- one solution,
- two solutions depending on the coefficients a, b, and c.

Factored form: $(x - x_1)(x - x_2) = 0$, where x_1 and x_2 are the solutions of the equation, also known as *roots.*

2. Solving the incomplete quadratic equation by factorizing.

- When $c = 0$, $ax^2 + bx = 0$

  $x(ax + b) = 0$ The two roots are $x_1 = 0$ and $x_2 = -b/a$
- When $b = 0$, $ax^2 + c = 0$

  If c < 0, factorize the equation using the formula for fast multiplication $a^2 - b^2 = (a-b)(a+b)$. (*)

For example, $x^2 - 25 = 0 \Rightarrow x^2 - 5^2 = 0 \Rightarrow (x - 5)(x + 5) = 0.$ $x = \pm 5$

If c > 0, there are no real solutions. An easy way to see this is to solve directly for x: $x^2 + 25 = 0 \Rightarrow x^2 = -5^2$; No number squared is equal to a negative number!

2. Solving the complete quadratic equation

- By completing the square

$$x^2 + 6x + 2 = x^2 + 2 \cdot 3x + 9 - 9 + 2 = (x+3)^2 - 7 = (x+3)^2 - \left(\sqrt{7}\right)^2 = \left(x + 3 + \sqrt{7}\right)\left(x + 3 - \sqrt{7}\right)$$

Thus, $x^2+6x+2 = 0$ if and only if $(x+3+\sqrt{7}) = 0$, which gives $x = -3-\sqrt{7}$, or $(x+3-\sqrt{7}) = 0$, which gives $x = -3+\sqrt{7}$.

- By using the quadratic formula

Completing the square works in general for any quadratic equation in a standard form

If $a = 1$, then:

$$x^2 + bx + c = x^2 + 2\frac{b}{2}x + c = \left(x^2 + 2\frac{b}{2}x + \frac{b^2}{2^2}\right) - \frac{b^2}{2^2} + c = \left(x + \frac{b}{2}\right)^2 - \frac{b^2 - 4c}{2^2} = \left(x + \frac{b}{2}\right)^2 - \frac{D}{2^2} \qquad \text{eq (1)}$$

Thus $x^2 + bx + c = 0$ is equivalent to: $\left(x + \frac{b}{2}\right)^2 = \frac{D}{4}$

If $a \neq 1$, then: $ax^2 + bx + c = 0$ divide by a $\Longrightarrow$ $\boldsymbol{x^2 + \frac{b}{a}x + \frac{c}{a} = 0}$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = \left(x^2 + 2\frac{b}{2a}x + \frac{b^2}{2^2a^2}\right) - \frac{b^2}{2^2a^2} + c = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{2^2a^2}$$

is equivalent to: $\left(x + \frac{b}{2a}\right)^2 = \frac{D}{4a^2}$, where $\boldsymbol{D = b^2 - 4ac}$

The determinant D determines the number of solutions. D < 0, there are no real solutions; if D = 0, there is one solution,

if $D > 0$, the solutions are: $x + \frac{b}{2a} = \pm\sqrt{\frac{D}{4a^2}}$

$$\boldsymbol{x = \frac{-b \pm \sqrt{D}}{2a}} \qquad \boldsymbol{D = b^2 - 4ac} \qquad \text{eq (2)}$$

Homework problems

Instructions: Please always write solutions on a ***separate sheet of paper***. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

Note: Use the formulas for fast multiplication $a^2 - b^2 = (a-b)(a+b)$, $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

1. Complete the square and solve the quadratic equations: using $(a \pm b)^2 = a^2 \pm 2ab + b^2$ $and\ then\ a^2 - b^2 = (a-b)(a+b)$
   a. $x^2 - 2x - 3 = 0$
   b. $x^2 + 8x - 9 = 0$

2. Solve the following equations. Carefully think what method you will use and write all steps in your solution. The following questions may help you: is the equation in a standard or in a factored form?; what are the coefficients a, b, c? Are some of these coefficients zero? Shall I factorize or use the quadratic formula from eq (2)?
   a. $x^2 - 5x + 5 = 0$
   b. $x^2 = 1 + x$
   c. $-4x^2 + 8x + 21 = 0$
   d. $2x(3 - x) = 1$
   e. $x^3 + 4x^2 - 45x = 0$
   f. $\frac{x}{x-2} = x - 1$

3. Indian mathematicians were aware of the quadratic formula for solving quadratic equations. Can you solve the following problem by the 9th century mathematician Mahavıra? (translated from original text)
   One-third of a herd of elephants and three times the square root of the remaining part of the herd were seen on a mountain slope; and in a lake was seen a male elephant along with three female elephants constituting the ultimate remainder. How many were the elephants here?

4. In the 12th century, Indian mathematician Bhaskara formulated the following problem. Solve it! (translated from original text)
   Out of a party of monkeys, the square of one fifth of their number diminished by three went into a cave. The one remaining monkey was climbing up a tree. What is the total number of monkeys?

5. Use eq (2) to solve these equations:
   a. $4x^2 - 58 + 5 = 0$
   b. $2x^2 + 5x + 3 = 0$

6. Determine the number of solutions of the following equations. You do not need to solve them!
   a. $2x^2 + 5x - 1 = 0$
   b. $3x^2 - 4x + 10 = 0$
   c. $3x^2 - 24x + 48 = 0$
   d. $5x^2 + 7x + 6 = 0$

7. If $x + \frac{1}{x} = 7$, find $x^2 + \frac{1}{x^2} = 7$ and $x^3 + \frac{1}{x^3}$ [Hint:try completing the square, completing the cube …]

8. Solve equation $x = \frac{x}{2} + \frac{1}{x}$; *[Hint: multiply by 2x]*