CHAPTER 3

CLEVER FACTORISATIONS. REVIEW ARITHMETIC SEQUENCES

3.1 Warm-Up: See Solved Homework questions

3.2 Diophantine Equations

One of the many applications of factoring, that goes back to the Greeks, is the solution of equations in which we seek only integer (sometimes positive integer) solutions. Such equations are called Diophantine equations. These sometimes use clever factorisations. **Example:** Find all positive integer solutions to $ab+2a+5b = 38$.
Solution

The first two terms factor as $a(b + 2)$. We add 10 to both sides, so that we can factor in pairs. Thus, $ab + 2a + 5b = 38 \Longleftrightarrow ab + 2a + 5b + 10 = 48 \Longleftrightarrow a(b + 2) + 5(b + 2) = 48 \Longleftrightarrow (a + 5)(b + 2) = 48$ Equating each bracket with a factor and cofactor of 48 and solving, we find that the only positive integer solutions are a = 11, b = 1; a = 1, b = 6; a = 3, b = 4; a = 7, b = 2.

3.3 Arithmetic Sequences

A sequence of numbers is an **arithmetic sequence** or **arithmetic progression** if the difference between consecutive terms is the same number, the **common difference** or d.
Example: The sequence $1, 5, 9, 13, 17, \ldots$ is an arithmetic sequence because the difference between consecutive terms is $d = 4$.

We can also find the n-th term if we know the 1st term and d?
Example: What is a_{100} in the example above?

$$
\begin{aligned}
&a_1 = 1\\
&a_2 = a_1 + d = 1 + 4 = 5\\
&a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d = (1 + 4) + 4 = 1 + 2 \times 4 = 9\\
&a_4 = a_3 + d = (a_2 + d) + d = ((a_1 + d) + d) + d = a_1 + 3d = 1 + 3 \times 4 = 13
\end{aligned}
$$

The pattern is:

$$
\begin{aligned}
a_n &= a_1 + (n - 1)d\\
a_{100} &= a_1 + 99d = 1 + 99 \times 4 = 397
\end{aligned}
$$

Properties of an Arithmetic Sequence

A useful property of an arithmetic sequence is that any term is the arithmetic mean of its neighbors:

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}$$

Proof:

$$a_n = a_{n-1} + d$$
$$a_n = a_{n+1} - d$$

Adding these two equalities gives us:

$$2a_n = a_{n-1} + a_{n+1}$$

from where we can get what we need.

Another property of arithmetic sequences is that we can find the common difference d if we know any two terms a_m and a_n:

$$d = \frac{a_m - a_n}{m - n}$$

Sum of an Arithmetic Sequence

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = n \times \frac{a_1 + a_n}{2}$$

Proof: To prove this, we write the sum in 2 ways, in increasing and decreasing order:

$$S_n = a_1 + a_2 + \cdots + a_n$$
$$S_n = a_n + a_{n-1} + \cdots + a_1$$

Adding these two expressions up and noticing that $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots$ we get:

$$2S_n = (a_1 + a_n) \times n$$
$$S_n = n \times \frac{a_1 + a_n}{2}$$

3.4 From Arithmetic Sequences To Linear Functions

Arithmetic sequences can be graphed as linear functions. While the n-value increases by a constant step of one, the f(n) value increases by the common difference d.

$$a_1 = 3 \times 1 + 2 = 5, a_2 = 3 \times 2 + 2 = 8, a_2 = 3 \times 3 + 2 = 11, \cdots$$

is the arithmetic sequence generated by the expression $3x + 2$

Homework

1. Write the first 5 terms of an arithmetic sequence if $a_1 = 7$ and $d = 2$.

2. What are the first 2 terms for the sequence

$$a_1, a_2, -9, -2, 5, \ldots?$$

3. $a_{10} = 131$ and $d = 12$. What is a_1?

4. $a_5 = 27$ and $a_{27} = 60$. Find the first term a_1 and the common difference d.
5. Write the linear expression/function that generates the given sequence 0, 11, 22, 33,....
6. Write the linear expression/function that generates the given sequence -10,-7,-4,-1,2,...
7. Write the sequence that is generated by the linear expression $4x + 7$
8. Find the common difference d in an arithmetic sequence if the 9-th term is 18 and the 11-th term is 44.
9. In the arithmetic progression $5, 17, 29, 41, \ldots$ what term has a value of 497?
10. Find the sum of the first 10 terms for the series: $4, 7, 10, 13, \ldots$
11. Find the sum of the first 1000 odd numbers.
12. Find the sum $2 + 4 + \cdots + 2018$.
13. In a given arithmetic progression, the first term is 6, and the 87-th term is 178. Find the common difference of this arithmetic progression, and give the value of the first five terms.
14. The 3-rd term of the arithmetic progression is equal to 1. The 10-th term of it is three times as much as the 6-th term. Find the first term and the common difference. (**Hint:** Use the formula for the n-th term of the progression and write what is given in the problem using this formula.)
15. There are 25 trees at equal distances of 5 meters in a line with a well, the distance of the well from the nearest tree being 10 meters. A gardener waters all trees separately starting from the well and he returns to the well after watering each tree to get water for the next. Find the total distance the gardener will cover in order to water all the trees.
16. An arithmetic progression has first term $a_1 = a$ and common difference $d = -1$. The sum of the first n terms is equal to the sum of the first $3n$ terms. Express a in terms of n.
17. Graph the first eight terms of the sequence $r_n = \frac{n^2}{n!}$. Describe how the value of r_n changes as n increases. Recall that $n! = n(n–1)(n - 2)...3 \times 2 \times 1$.

18* The sum of the first 20 terms of an arithmetic progression is 200, and the sum of the next 20 terms is -200. Find the sum of the first hundred terms of the progression.