CHAPTER 4

REVIEW GEOMETRIC SEQUENCES

Other approaches for arithmetic sequences – graphical proof by induction

Given some n, can we compute the sum of odd numbers up to $2n-1$ in a different way than just applying the formula ? Yes, if we use "square tokens", lining them up themselves in a growing square pattern:

```
 1
 _
|_|


 1+3 = 4
 _ _
|_|_|
|_|_|


 (1+3)+5 = (1+3)+(2+3) = (1+3)+(2+2+1) = 9
 _ _ _
|_|_|_|
|_|_|_|
|_|_|_|


 ((1+3)+5)+7 = 9 + (3 + 4) = 9 + (3 + 3 + 1) = 16
 _ _ _ _
|_|_|_|_|
|_|_|_|_|
|_|_|_|_|
|_|_|_|_|

...
```

Geometric sequences

A sequence of numbers is a **geometric sequence** or **geometric progression** if if the next number in the sequence is the current number times a fixed constant called the **common ratio** or q.
Example: The sequence $6, 12, 24, 48, \dots$ is a geometric sequence because the next number is obtained from the previous by multiplication by $q = 2$.

We can also find the n-th term if we know the 1st term and q.
Example: What is a_{10} in the example above?

$$\begin{aligned} a_1 &= 6 \\ a_2 &= a_1 q = 6 \cdot 2 = 12 \\ a_3 &= a_2 q = (a_1 q) q = a_1 q^2 = 6 \cdot 2^2 = 24 \end{aligned}$$

The pattern is:

$$\begin{aligned} a_n &= a_1 q^{n-1} \\ a_{10} &= a_1 q^9 = 6 \cdot 2^9 = 6 \cdot 512 = 3072 \end{aligned}$$

Properties of a Geometric Sequence

Any term is the **geometric mean** of its neighbors:

$$a_n = \sqrt{a_{n-1} \cdot a_{n+1}}$$

Proof:

$$\begin{aligned} a_n &= a_{n-1} q \\ a_n &= a_{n+1} / q \end{aligned}$$

Multiplying these two equalities gives us:

$$a_n^2 = a_{n-1} \cdot a_{n+1}$$

from where we can get what we need.

Sum of a Geometric Sequence

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \frac{a_1(1 - q^n)}{1 - q}$$

Proof: To prove this, we write the sum and multiply it by q:

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_n \\ qS_n &= qa_1 + qa_2 + \cdots + qa_n \end{aligned}$$

Now notice that $qa_1 = a_2, \dots qa_2 = a_3, \dots, qa_n = a_{n+1}$, etc, so we have:

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_n \\ qS_n &= a_2 + a_3 + \cdots + a_{n+1} \end{aligned}$$

Subtracting the second equality from the first, and canceling out the terms, we get:

$$\begin{aligned} S_n - qS_n &= (a_1 - a_{n+1}), \text{ or} \\ S_n(1 - q) &= (a_1 - a_1 q^n) \\ S_n(1 - q) &= a_1(1 - q^n) \end{aligned}$$

from which we get the formula above.

Infinite Sum

If $0 < q < 1$, then the sum of the geometric progression is approaching some numbers, which we can call a **sum of an infinite geometric progression**, or just an **infinite sum**.

For example:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2.$$

The formula for the infinite sum is the following:

$$S = \frac{a_1}{1-q}$$

Homework

1. Write the first 5 terms of a geometric progression if $a_1 = -20$ and $q = 1/2$.
2. What are the first two terms of the geometric progressions $a_1, a_2, 24, 36, 54, \ldots$?
3. Find the common ratio of the geometric progressions $1/2, -1/2, 1/2, \ldots$. What is a_{10}?
4. Calculate:
$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^{10}}.$$
5. Calculate:
$$1 - 2 + 2^2 - 2^3 + 2^4 - 2^5 + \cdots - 2^{15}$$
6. Calculate:
$$1 + x + x^2 + x^3 + x^4 + \cdots + x^{100}$$
7. A geometric progression has 99 terms, the first term is 12 and the last term is 48. What is the 50th term?
8. If we put one grain of wheat on the first square of a chessboard, two on the second, then four, eight, ..., approximately how many grains of wheat will there be? (you can use an approximation $2^{10} = 1024 \approx 10^3$).

   Can you estimate the total volume of all this wheat and compare with the annual wheat harvest of the US, which is about 2 billion bushels. (A grain of wheat is about 10 mm^3; a bushel is about 35 liters, or 0.035 m^3)
9. Musicians use special notations for notes, i.e. sound frequencies. Namely, they go as follows:

$$\ldots,\ \text{A},\ \text{A}\sharp,\ \text{B},\ \text{C},\ \text{C}\sharp,\ \text{D},\ \text{D}\sharp,\ \text{E},\ \text{F},\ \text{F}\sharp,\ \text{G},\ \text{G}\sharp,\ \text{A},\ \text{A}\sharp,\ \ldots$$

   The interval between two notes in this list is called a **halftone**; the interval between A and the next A (or B and next B, etc.) is called an **octave**. Thus, one octave is 12 halftones. (If you have never seen it, read the description of how it works in Wikipedia.)

   It turns out that the frequencies of the notes above form a geometric (not an arithmetic!!) sequence: if the frequency, say, of A in one octave is 440 hz, then the frequency of A$\sharp$ is $440r$, frequency of B is $440r^2$, and so on.

   (a) It is known that moving by one octave doubles the frequency: if the frequency of A in one octave is 440 hz, then the frequency of A in the next octave is $2 \times 440 = 880$ hz. Based on that, can you find the common ratio r of this geometric sequence?

(b) Using the calculator, find the ratio of frequencies of A and E (such an interval is called a **fifth**). How close is it to 3 : 2?

Historic reference: the above convention for note frequencies is known as "equal temperament" and was first invented around 1585. However, it was not universally adopted until the beginning of 19th century. One of the early adopters of this tuning method was J.-S. Bach, who composed in 1722–1742 a collection of 48 piano pieces for so tuned instruments, called Well-Tempered Clavier. Find them and enjoy! If you want to know how musical instruments were tuned before that, do your own research.