CHAPTER 8

PERMUTATIONS. COMBINATIONS. PASCAL TRIANGLE

8.1 Recall: Fundamental Principle of Counting. Permutations. Combinations.

Fundamental Principle of Counting (Multiplication rule)

If a task can be performed in m ways, and for each of these a second task can be performed in n ways, and for each of the latter a third task can be performed in k ways, then the sequence of k tasks can be performed in $m \cdot n \cdot k$ ways.

Permutation

The choice of k things from a set of n things without replacement and where the order matters is called a *permutation*. Examples:

1. Picking first, second and third place winners from a group.
2. Listing the favorite deserts in the order of choices.

Particular case: **Permutations of n things:** The number of permutations of n different things taken n at a time: $n!$

Combination

The choice of k things from a set of n things without replacement and where order does not matter is called a *combination*. Examples:

1. Picking three team members from a group.
2. Picking two deserts from a tray.

8.2 Pascal's Triangle

Motivational Problem Calculation of Combinations: Consider a grid that has 5 rows of 5 squares in each row with the lower left corner named A and upper right corner named B. Suppose that starting at point A you can go one step up or one step to the right at each move. This is continued until the point B is reached. How many different paths from A to B are possible?

We can graphically solve small cases, even up to 6x6 – see figures

A more general strategy for the Case 4x4

If we start at A and move towards B, we find we can follow the path RRRRUUUU (where R = Right one unit, U = Up one unit), UUUURRRR, RURURURU, RRUUUURR, and so on.

By analyzing our good routes, we see that every good route consists of eight moves and we have four R moves and four U moves. Since we can have an R in any place of our eight moves, and we only have two choices for moves: right and up, the number of how many good routes we have can be found by finding how many combinations of four R's we can have in our eight moves. (for example R:5-th move, R:6-th move, R-7th move, R:8th move). The U's will then be the remaining four unassigned moves.

If we simply enumerate all possibilities, we can say that we have eight choices for the first 'right move' R to place in our sequence of eight moves. In other words a first R can be any of these eight moves. Then we are left with seven unassigned moves, so one of them can be another R. Then we are left with six unassigned moves, so one of them can be a third R. Finally, for the fourth and last R to assign, we are only left with five unassigned possible moves. This gives an initial total of $8 \cdot 7 \cdot 6 \cdot 5 = \frac{8!}{4!}$ total possibilities, at first sight.

However, counting in this way we end up counting more than once the same "route", because '$\mathrm{R}_1 U \mathrm{R}_2 \ldots$ is the same as $\mathrm{R}_2 U \mathrm{R}_1 \ldots$, when the indices 1, 2, etc denote *when we decided to pick them.* What matters is when we *walk through* them, that is whether the first move (out of the eight) is an 'R' or an 'U', and not whether we first assigned an 'R' to the third move and then decided that also the first one would be an 'R' or the opposite sequence of *decisions*. The "final product" is the same in both cases.

All this means that different orderings of the various 'R's in our enumeration must be deemed the same. How many possible orderings can there be with 4 elements ? That's the permutation of 4 out of 4, so 4! and that's what we need to divide the initially obtained number by.

So the correct number of ways to get from A to B in the 4x4 square with R and U is $\frac{8!}{4! \cdot 4!}$.

This reasoning is of course valid for any grid, even rectangular ones. Take some positive integer n and some positive integer k with k at most n. Consider the same A-to-B route counting in a $(n-k) \times k$ grid, that is with k columns and $n-k$ rows. Then we need n moves (from $k + (n-k)$), out of which k are 'R'.

We thus obtain what is called in general the number of combinations of n things taken k at a time (called "[[n choose k]]"), i.e.

$$\mathbf{C}(n,k) = \mathbf{C}^n_k = {}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

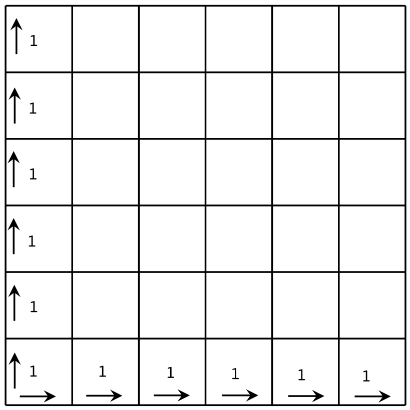
In our particular case ${}_8C_4$=70.

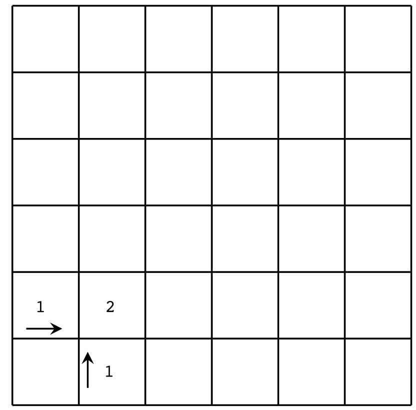
Can we do it iteratively?

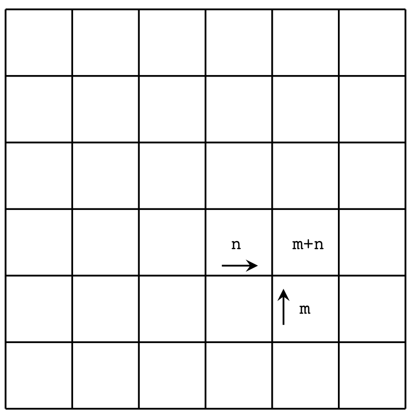
The rules allow moving only up or right. So the total number of routes to an inner grid point would be equal to the sum of routes to its left point and routes to the point right below it, as there is only one route from left point and only one route from lower point. Furthermore, a route is possible only from left point or lower point but not both. If there is no left point, as is the case for all points on the left most vertical line, then the number of routes is same as the number of routes to the lower point. Likewise, if there is no lower point, as is the case for all points on the lower most horizontal line, then the number of routes to the point is same as the number of routes to the left point. These are illustrated in the following set of diagrams.

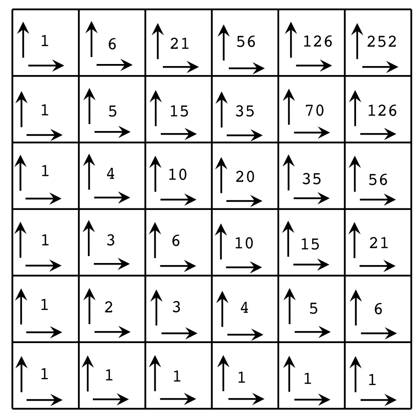
Interesting combinatorics results to be proven graphically:

The main point in the argument is that each entry in row n, say ${}_nC_k$ is added to two entries below: once to form ${}_{n+1}C_k$ and once to form ${}_{n+1}C_{k+1}$ which follows from Pascal's Identity:









$$ {}_{n+1}C_k = {}_nC_{k-1} + {}_nC_k, \; {}_{n+1}C_{k+1} = {}_nC_k + {}_nC_{k+1} $$

Pascal's triangle determines the coefficients which arise in binomial expansions

8.3 Problems

1. A dinner in a restaurant consists of 3 courses: appetizer, main course, and dessert. There are 5 possible appetizers, 6 main courses and 3 desserts. How many possible dinners are there?

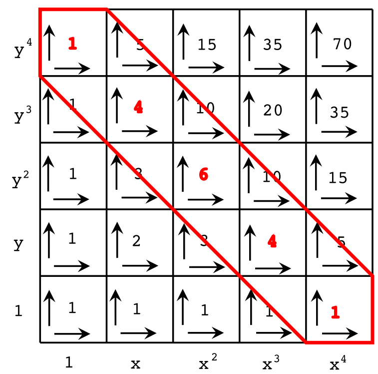
2. How many ways are there to select first, second and third prize winner if there are 14 athletes in a competition?

3. How many ways are there to seat 5 students in a class that has 5 desks? if there are 10 desks?

4. How many ways are there to put 8 rooks on a the chessboard so that no one attacks the others?

5. How many different committees of 5 students can be chosen from a group of 8?

6. A dressmaker has two display windows. The left display is for evening dresses and the one in the right window for regular day dresses. Assuming she can put the 10 evening dresses in any order, and separately, the five regular dresses in any order, how many total possibilities of arranging the two



display windows are there? (Hint: Any of the displays of the left window could be combined with any of the displays of the right window)

7. The county-mandated guidelines at a certain community college specify that for the introductory English class, the professor may choose one of three specified novels, and choose two from a list of 5 specified plays. Thus, the reading list for this introductory class is guaranteed to have one novel and two plays. How many different reading lists could a professor create within these parameters?

8. How many ways are there to go from the bottom left corner of the chessboard to the upper right, moving always only to the right and up?

9. Let T_n be the number of circles in nth figure (for example, $T_1 = 1$, $T_2 = 3$, $T_3 = 6...$). It is called a triangular number. More formally, the triangular number T_n is a figurate number that can be represented in the form of a triangular grid of points where the first row contains a single element and each subsequent row contains one more element than the previous one.

   (a) Algebraically $T_n = \sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} = \binom{n+1}{2}$. Can you prove it graphically using the Pascal triangle?

   (b) What is the difference $T_{n+1} - T_n$? (Hint: graphically and algebraically the sum of two consecutive triangular numbers is a square number)

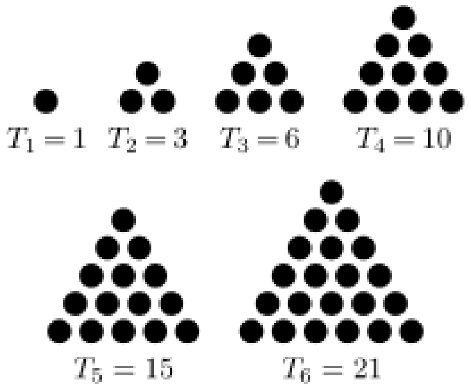
   (c) $T_{n+m} = T_n + T_m + nm$

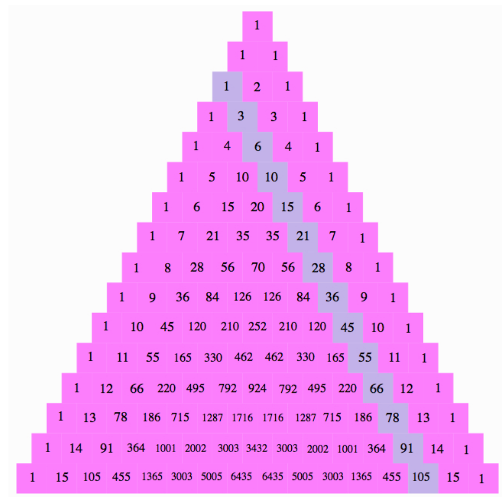
   (d) Check that the triangular numbers T_n appear in the Pascal triangle

10. Which of the numbers in Pascal triangle are even? Can you guess the pattern, and then carefully explain why it works?

11. What is the sum of all entries in the nth row of Pascal triangle? Try computing first several answers and then guess the general formula.

12. What is the alternating sum of all the numbers in nth row of Pascal triangle, i.e. $1 - {}_nC_1 + {}_nC_2 - {}_nC_3 + \ldots$


$T_1 = 1$
$T_2 = 3$
$T_3 = 6$
$T_4 = 10$
$T_5 = 15$
$T_6 = 21$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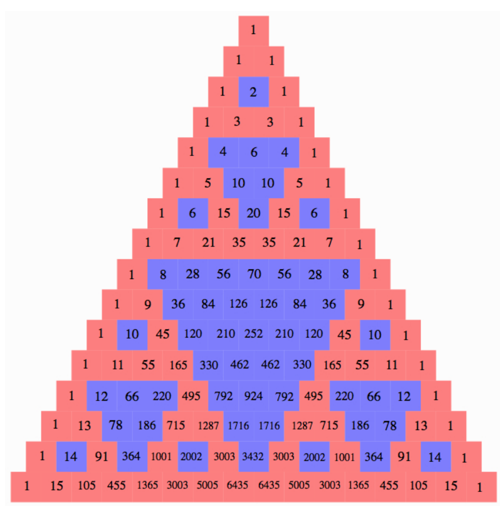
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