CHAPTER 15

GRAPHING QUADRATIC POLYNOMIAL FUNCTIONS

15.1 Shape of quadratic graphs and end behavior of polynomial functions

Recall: End behavior of a polynomial. Axes of Symmetry

We have two cases x becomes larger, and larger in the positive direction, denoted by $x \to \infty$, and x becomes larger, and larger in the negative direction, denoted by $x \to -\infty$

Examples

- x^2, $-x^2$, x^4, $-x^4$ (identify points/axes of symmetry)
- x^3, $-x^3$, x^5, $-x^5$ (identify points/axes of symmetry)

In general, the end behavior is determined by the term that contains the highest power of x, called the leading term. Why? Because when x is large, all the other terms are small compared to the leading term.

What happens in the middle of the graph of a quadratic function?

In some locations the graph behavior changes. A turning point is a point at which the function values change from increasing to decreasing or decreasing to increasing.

Method 1

Due to the symmetry of the parabola, the turning point lies halfway between the x-intercepts. As a consequence, if there is only one x-intercept, then the x-intercept is exactly the turning point.

Examples:

x^2, $x^2 + 4x + 4$, $x^2 + 3x + 2$

Method 2

The first method works if the x-intercepts exist. However, if we re-write the quadratic polynomial function as we did in the "completing the square" we have

$$y = f(x) = a(x - h)^2 + k \text{ , and its turning point is } (x, y) = (h, k)$$

or more precisely

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a} \text{ , and its turning point is } (x, y) = \left(-\frac{b}{2a}, -\frac{D}{4a}\right)$$

Examples:

x^2, x^2+2x+4, x^2+3x+2

Axis of symmetry

The axis of symmetry is a useful line to find since it gives the x-coordinate of the vertex of the parabola which is its turning point discussed above. We can complete the square on the general quadratic polynomial function and thereby obtain a general formula for the axis of symmetry and hence the x-coordinate for the vertex.

$$a\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a}\text{ , and its vertex is }(x,y)=\left(-\frac{b}{2a},-\frac{D}{4a}\right)$$

This expression shows that the minimum (or maximum in the case when a is negative) occurs when the first bracket is zero, that is, when $x=-\frac{b}{2a}$.

15.2 Classwork

Given the polynomial functions

1. $f(x)=(x-1)(x+2)(x-3)$,
2. $f(x)=(x-4)(x+5)$
3. $f(x)=(x-4)^2$

express the function as a polynomial in general form and determine the leading term, degree, and its end behavior. Determine the turning point/ parabola vertex for the last two functions and try to sketch their graph.

15.3 Sketch a graph of a quadratic polynomial function using its zeroes

Find the roots:

$$y=f(x)=x^2+x-2$$

Find the roots of the polynomial function of second degree (that is, the zeros of the quadratic equation):

$$f(x)=(x+2)(x-1)$$

The roots are x-intercepts: $(-2,0)$, $(1,0)$.

Axis of symmetry and turning point:

Halfway between the x-intercepts:

$$x_t=-\frac{b}{2a}=-\frac{1}{2}=-0.5$$

$$\Longrightarrow y_t=(-0.5)^2-0.5-2=-0.25-2=-2.25$$

End behavior:

$$x \to \infty \Longrightarrow f(x) \to \infty, f(x) > 0$$

$$x \to -\infty \Longrightarrow f(x) \to \infty, f(x) > 0$$

x	$-\infty$		-2		-0.5		$+1$		$+\infty$
$(x+2)$									
$(x-1)$									
$f(x) = (x+2)(x-1)$									

We take test points in these intervals to determine the signs.
For example, test point for

$$\begin{array}{lll} (-\infty, -2) & : & x = -3 \\ (-2, -0.5) & : & x = -1 \\ (-0.5, 1) & : & x = 0 \\ (1, \infty) & : & x = 2 \end{array} \Longrightarrow$$

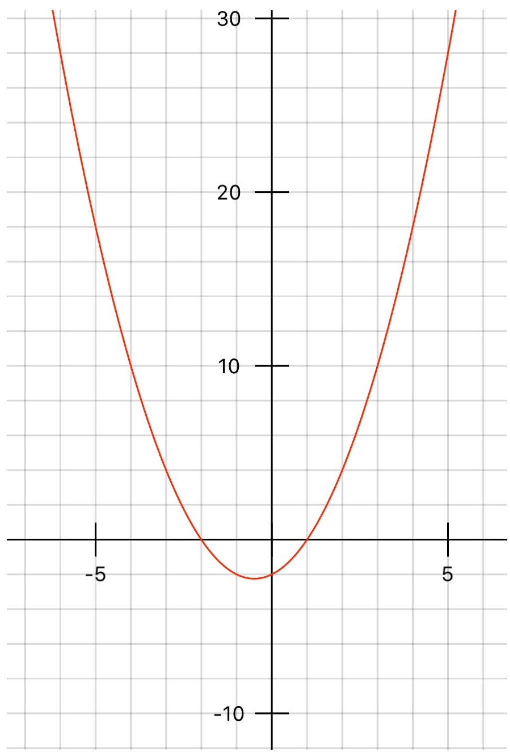

Figure 15.1: Graph of $f(x) = (x+2)(x-1)$

15.4 Worked out examples

Examples for turning point/ parabola vertex:

x^2, $x^2 + 4x + 4$, $x^2 + 3x + 2$

Method 1: Symmetry

- The x-intercept is found by letting $y = f(x) = 0$. So, $0 = x^2 \implies x = 0$. There is only one x-intercept, so that gives the x-coordinate of the turning point. To find the y-coordinate, substitute it into the quadratic equation. $f(0) = 0^2 \implies y = 0$. Thus, the turning point is (x, y) = (0, 0).
- $x^2+4x+4 = (x+2)^2$ There is only one x-intercept, so that gives the x-coordinate of the turning point.
- $x^2+3x+2 = (x+1)(x+2)$. So, there are two roots, thus two distinct x-intercepts: $x = -1$ or $x = -2$. To find the x-coordinate of the turning point, average the x-intercepts. So, $x_{tp} = \frac{(-1)+(-2)}{2} = -\frac{3}{2}$. To find its y-coordinate, substitute this value into the equation: $y_{tp} = \left(-\frac{3}{2}\right)^2 + 3\left(-\frac{3}{2}\right) + 2 = \frac{9}{4} - \frac{9}{2} + 2 = -\frac{1}{4}$. Thus, the coordinates of the turning points are $(x, y) = \left(-\frac{3}{2}, -\frac{1}{4}\right)$.

Method 2: Completing to Square, Discriminant Formula

- For x^2, if we complete the square, we obtain: $(x-0)^2 + 0$. So, its turning point is $(x, y) = (0, 0)$.
- For x^2+3x+2, if we complete the square, we obtain: $y = x^2+3x+\left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 2 = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + 2 = \left[x - \left(-\frac{3}{2}\right)\right]^2 - \frac{1}{4}$. So, the turning point is $(x, y) = \left(-\frac{3}{2}, -\frac{1}{4}\right)$.

Example of using the zeroes to graph a quadratic polynomial function

x	$-\infty$	-3	-2	-1	-0.5	0	$+1$	$+2$	$+\infty$
$(x+2)$	$-\infty$	-1	0	$+1$	$+1.5$	$+2$	$+3$	$+4$	$+\infty$
$(x-1)$	$-\infty$	-4	-3	-2	-1.5	-1	0	$+1$	$+\infty$
$f(x) = (x+2)(x-1)$	$+\infty$	$+4$	0	-2	-2.25	-2	0	$+4$	$+\infty$

15.5 Homework

- For what values of "b" has the polynomial function $x^2 + bx + 14$ no real numbers roots? exactly one root? two distinct real numbers roots?
- Solve the following inequalities using a table similar to one used for graphing quadratic functions:

  1. $x^2 - x + 6 \geq 0$
  2. $\frac{2x+1}{x-5} \leq 0$
  3. $x(x-2)(x+18) \geq 0$
  4. $x^2 + x \geq 0$
  5. $x^2 - 5x + 6 \leq 0$
  6. $|x^2 - x| > 1$

- Sketch the graphs of the following functions and relations:

  1. $x + y = 1$
  2. $y = x^2 + x$
  3. $y = x^2 - 5x + 6$
  4. $y = (x-2)(x+18)$
  5. $|x + y| = 4$
  6. $y = |x+1| - |x-1|$
  7. $y = |x^2 - x|$
  8. $x^2 + 4x + y^2 - 4y = 0$