CHAPTER 16

ABSOLUTE VALUES, EQUATIONS AND INEQUALITIES

16.1 Homework

16.1.1 Lines and parabolas

Recall that the graph of $y = a \cdot x + b$ is a straight line, and its slope depends on a and in particular on its sign. The line will pass through zero only if $b = 0$. Also, review the previous handout for the parabolas and the table with signs and roots, to do the following.

1. Find the x-intercept and the y-intercept and plot the following:
   (a) $f_1(x) = x + 3$
   (b) $f_2(x) = 2 - x$
2. Write the table with signs and roots as we did in class, and plot the following parabolas:
   (a) $g_1(x) = (x - 2) \cdot (x + 3)$
   (b) $g_2(x) = -(x + 1) \cdot (x + 2)$
3. Solve the following inequalities, using again the table with signs and roots. Also show the solution on the real line, and write the answer in the interval notation.
   (a) $(x - 1)(x + 2) > 0$
   (b) $(x + 3)(x - 2)^2 \leq 0$
   (c) $x(x - 1)(x + 2) \geq 0$
   (d) $x^2(x + 1)^5(x + 2)^3 > 0$
4. Solve the following equations:
   (a) $\dfrac{x + 1}{x - 1} = 3$
   (b) $\dfrac{x^2 - 9}{x + 1} = x + 3$
   (c) $x - \dfrac{3}{x} = \dfrac{5}{x} - x$

16.1.2 Absolute values

Recall that $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x < 0$. So, for instance, $|x + 1|$ is going to be $x + 1$ when $x + 1 \geq 0$ which means $x \geq -1$. This allows us to plot functions and solve equations and inequations. So, for instance, the graph of $y = |x|$ is made of two lines, one for each *region*: from 0 inclusive to $+\infty$ the graph is the line $y = x$, and from $-\infty$ to 0 exclusive the graph is the line $y = -x$. Also, solving for instance $|x + 1| = 2$ is done by analyzing each of these cases: so we suppose $x + 1 \geq 0$ which means $x \geq -1$, and then the equation becomes $x + 1 = 2$ which means $x = 1$. We verify the condition $x \geq -1$, which is respected, so that is one solution: $x_1 = 1$. Then, we suppose $x + 1 < 0$ which means $x < -1$, and the equation becomes $-(x + 1) = 2$ which leads to $-x - 1 = 2$ which means $x = -3$, which also respects $x < -1$, the condition for this case. Therefore there are two solutions: $x_1 = 1$ and $x_2 = -3$.

Finally, solving $|x + 1| > 2$ is done exactly in the same way, by analyzing the same two cases. So keep in mind that the cases are to be analyzed for each absolute value, for the respective quantity, and then solutions need to be checked to also respect the case requirement, as above. For inequalities, the solution intervals need to be intersected with the case requirement intervals (e.g. $x \geq -1$ to be able to claim that $|x + 1|$ is in fact $x + 1$).

These rules are to be applied regardless of the expression we are taking the absolute value of. So in general, $|h(x)|$ is going to be $h(x)$ whenever $h(x) \geq 0$ and $-h(x)$ when $h(x) < 0$. In particular, $|a \cdot x^2 + b \cdot x + c|$ needs to be analyzed the way we would analyze the quadratic itself, when comparing it with zero.

1. Plot the following:
   (a) $f_3(x) = |x|$
   (b) $f_4(x) = |x - 1|$
   (c) $f_5(x) = |x - 1|$
2. Solve the following equations:
   (a) $|x| = 4$
   (b) $|x - 3| = 5$
   (c) $|2x - 1| = 7$
   (d) $|x^2 - 5| = 4$
3. Solve the following inequalities. Also show the solution on the real line, and write the answer in the interval notation. Remember to verify that we are not dividing by zero as needed.
   (a) $|x - 2| > 3$
   (b) $|x - 1| > x + 3$
   (c) $\dfrac{x - 2}{x + 3} \leq 3$