MATH 7
ASSIGNMENT 3: LINEAR AND QUADRATIC EQUATIONS
OCT 17, 2021

The simplest types of algebraic equations are linear and quadratic equations. You will review/practice solving linear equations and systems of linear equations in your homework. We also introduce a first method to solve quadratic equations: using Vieta's formulas.

1. **Linear Equations**

Any equation of the form $ax + b = 0$ has a simple, unique solution:

$$x = -\frac{b}{a}.$$

2. **Systems of Linear Equations**

Sometimes we have two variables which should satisfy two equations simultaneously. This is called a *system of linear equations*:

$$ax + by = c \tag{1}$$

$$dx + ey = f, \tag{2}$$

where the variables are x and y. One can solve it by *substitution*: from equation (1),

$$y = \frac{c - ax}{b}.$$

substituting this equation (2), we get a linear equation!,

$$dx + e\left(\frac{c - ax}{b}\right) = f \Rightarrow \left(d - \frac{ea}{b}\right)x + \left(\frac{ec}{b} - f\right) = 0.$$

One can solve this for x, and then use that answer to find y.

3. **Vieta formulas**

If an equation $p(x) = 0$ has root x_1 (i.e., if $p(x_1) = 0$), then $p(x)$ is divisible by $(x - x_1)$, i.e. $p(x) = (x-x_1)q(x)$ for some polynomial $q(x)$. In particular, if $x_1; x_2$ are roots of quadratic equation $ax^2+bx+c = 0$, then $ax^2 + bx + c = a(x - x_1)(x - x_2)$. Therefore,

$$\begin{aligned} x_1 + x_2 &= -\frac{b}{a} \\ x_1x_2 &= \frac{c}{a} \end{aligned}$$

These formulas are called *Vieta Formulas.* They can be used to find the solutions of a quadratic equation.

Homework

1. Solve the following equations:
   (a) $\frac{x+3}{x+1} = 4$
   (b) $2x + 25 = 5x + 10$
   (c) $\frac{x}{2} + 1 = \frac{4x}{7}$
   (d) $x = \frac{x}{4} + 6$
   (e) $x + 2(x - 5) = \frac{1}{2}(x + 3)$

2. The pet store sells parrots and canaries. A canary costs twice as much as a parrot. One customer bought 5 canaries and 3 parrots, while the other bought 3 canaries and 5 parrots. One of the customers paid \$20 more than the other. How much does each bird cost?

3. The teacher asked the students to multiply a given number by 4 and then add 15. However, one of the students multiplied the number by 15 and then added 4 — and still got the correct answer. What number was it?

4. Solve the following systems of equations

(a)

$$\begin{aligned} x &= 5 \\ 20x + 5y &= 100 \end{aligned}$$

(b)

$$\begin{aligned} -8x + y &= -4 \\ -21x + 2y &= -13 \end{aligned}$$

(c)

$$\begin{aligned} 7x - 3y &= 27 \\ 5x - 6y &= 0 \end{aligned}$$

(d)

$$\begin{aligned} 2(x-2) - 3(x+y) &= 3 \\ (x+1)(y-2) &= xy - 9 \end{aligned}$$

(e)

$$\begin{aligned} \frac{2x-1}{5} + \frac{3y-2}{4} &= 2 \\ \frac{3x+1}{5} - \frac{3y+2}{4} &= 0 \end{aligned}$$

5. Let a and b be some numbers. Use the formulas discussed in previous classes to express the following expressions using only $(a+b) = x$ and $ab = y$.

Example: Let's express $a^2 + b^2$ using only $a + b$ and ab. We know that $(a+b)^2 = a^2 + 2ab + b^2$. From here, we get:

$$a^2 + b^2 = (a+b)^2 - 2 \times ab = x^2 - 2 \times y$$

(a) $(a-b)^2$
(b) $\frac{1}{a} + \frac{1}{b}$
(c) $a - b$
(d) $a^2 - b^2$
(e) $a^3 + b^3$ (Hint: first compute $(a+b)(a^2+b^2)$)

6. Let x_1, x_2 be roots of the equation $x^2 + 5x - 7 = 0$. Find
(a) $x_1^2 + x_2^2$
(b) $(x_1 - x_2)^2$
(c) $\frac{1}{x_1} + \frac{1}{x_2}$
(d) $x_1^3 + x_2^3$

7. Solve the following equations:
(a) $x^2 - 5x + 6 = 0$
(b) $x^2 = 1 + x$
(c) $\sqrt{2x+1} = x$
(d) $x + \frac{1}{x} = 3$

8. Solve the equation $x^4 - 3x^2 + 2 = 0$

9. (a) Prove that for any $a > 0$, we have $a + \frac{1}{a} \geq 2$, with equality only when $a = 1$.
(b) Show that for any $a, b \geq 0$, one has $\frac{a+b}{2} \geq \sqrt{ab}$. (The left hand side is usually called the *arithmetic mean* of a, b; the right hand side is called the *geometric mean* of a, b.)