MATH 7
ASSIGNMENT 4: QUADRATIC EQUATIONS AND INEQUALITIES
OCT 31, 2021

Today we will go beyond equations to linear and quadratic inequalities. Before that, let us practice a more general method for solving quadratic equations.

1. Quadratic Equations: Completing the Square

"Completing the square" is an alternative method to solve quadratic equations, which in general have the form

$$ax^2 + bx + c = 0.$$

Here is an example of how it works:

$$x^2 + 6x + 2 = x^2 + 2 \cdot 3x + 9 - 7 = (x+3)^2 - 7 = (x + 3 + \sqrt{7})(x + 3 - \sqrt{7})$$

thus, $x^2 + 6x + 2 = 0$ if and only if $x + 3 + \sqrt{7} = 0$, which gives $x = -3 - \sqrt{7}$, or $x + 3 - \sqrt{7} = 0$, which gives $x = -3 + \sqrt{7}$.

The same trick works in general: if $a = 1$, then

$$\begin{aligned} x^2 + bx + c &= x^2 + 2\tfrac{b}{2}x + c = \left(x^2 + 2\tfrac{b}{2}x + \tfrac{b^2}{2^2}\right) - \tfrac{b^2}{2^2} + c \\ &= \left(x + \tfrac{b}{2}\right)^2 - \tfrac{b^2 - 4c}{4} = \left(x + \tfrac{b}{2}\right)^2 - \tfrac{D}{4} \end{aligned} \tag{1}$$

where $D = b^2 - 4c$.

Thus, $x^2 + bx + c = 0$ is equivalent to

$$\left(x + \tfrac{b}{2}\right)^2 = \tfrac{D}{4}$$

If a is not equal to 1, the answer is similar: $ax^2 + bx + c = 0$ is equivalent to

$$\left(x + \tfrac{b}{2a}\right)^2 = \tfrac{D}{4a^2}, \qquad D = b^2 - 4ac$$

Therefore, if $D < 0$, there are no solutions; if $D \geq 0$, solutions are

$$\begin{aligned} x + \tfrac{b}{2a} &= \pm\frac{\sqrt{D}}{2a} \\ x &= \frac{-b \pm \sqrt{D}}{2a} \end{aligned} \tag{2}$$

2. Solving Quadratic Inequalities

As we saw in class, if you know how to solve linear inequalities, you're not far from knowing how to solve quadratic inequalities. Here is a summary of the steps:

- Find the roots and factor your polynomial, writing it in the form $p(x) = a(x - x_1)(x - x_2)$.
- Roots $x_1, x_2, \ldots$ divide the real line into intervals; define the sign of each factor and the product on each of the sign intervals.
- If the inequality has $\geq$ or $\leq$ signs you should also include the roots themselves into the intervals.

2.1. Examples.

1. $x^2 + x - 2 > 0$. We find roots of the equation $x^2 + x - 2 = 0$ and obtain $x = -2, 1$. The inequality becomes $(x + 2)(x - 1) > 0$ and roots -2, 1 divide the real line into three intervals $(-\infty, -2), (-2, 1), (1, +\infty)$. It is easy to see that the polynomial $x^2 + x - 2$ is positive on the first and the third intervals and negative on the second one. The solution of the inequality is then $x < -2$ or $x > 1$. We sometimes, write this also as $x \in (-\infty, -2) \cup (1, +\infty)$. (sign $\cup$ means "or").

2. $-x^2 - x + 2 \geq 0$. We have $-(x + 2)(x - 1) \geq 0$. The left hand side is positive for $-2 < x < 1$. As the sign in the inequality is $\geq$ we have to includes the roots into the interval and obtain $-2 \leq x \leq 1$. One can also write $x \in [-2, 1]$ (square brackets here mean that the endpoints of the interval are included).

3. $x^2+x+2 \geq 0$. The polynomial here does not have roots (the discriminant $12-4\cdot1\cdot2<0$). Therefore, the real line is not divided into the intervals, which means that the polynomial is of the same sign for all x. We check that it is positive, for example, for $x=0$. The solution is that x is any number. We can write $x \in (-\infty, +\infty)$.

4. $x^2+x+2<0$. The polynomial does not have roots and is positive everywhere. This means that the inequality does not have solutions at all. One can also write $x \in \emptyset$.

5. $x^2-2x+1>0$. The inequality is $(x-1)^2>0$. There is only one root here which divides the real line into two intervals. The solution is $x<1$ or $x>1$, that is any x except for $x=1$. One can write $x \in (-\infty, 1) \cup (1, +\infty)$.

Homework

1. Without solving the equation $3x^2-5x+1=0$ find the arithmetic mean of its roots (that is $\frac{x_1+x_2}{2}$) and their geometric mean (that is $\sqrt{x_1x_2}$).

2. Find the roots of the equation $4x^2-2x-1=0$ **WITHOUT** using the formula for roots of quadratic equation. That is, complete the square and use the difference of squares formula to factorize the polynomial.

3. Solve the following equations. Carefully write all the steps in your argument.
  (a) $x^2-5x+5=0$
  (b) $\frac{x}{x-2}=x-1$
  (c) $x^2=1+x$
  (d) $2x(3-x)=1$
  (e) $x^3+4x^2-45x=0$

4. Solve the following inequalities:
  (a) $\frac{x+3}{x+1}>4$
  (b) $2x+25<5x+10$
  (c) $\frac{x}{2}+1 \geq \frac{4x}{7}$
  (d) $x \leq \frac{x}{4}+6$

5. (a) Use formula (1) to prove that for any x, $x^2+bx+c \geq -D/4$, with equality only if $x=-b/2$.
  (b) Find the minimal possible value of the expression x^2+4x+2 [Hint: use the result from part (a)]
  (c) Given a number $a>0$, find the maximal possible value of $x(a-x)$ (the answer will depend on a). [Hint: use the result from part (a)]

6. Solve the equation $x^4-x^2-2=0$.

7. Solve the following equations and inequalities:
  (a) $x^2+2x-3=0, \quad x^2+2x-3>0, \quad x^2+2x-3 \leq 0$
  (b) $x^2+2x+3=0, \quad x^2+2x+3 \geq 0, \quad x^2+2x+3<0$
  (c) $-x^2+6x-9=0, \quad -x^2+6x-9 \geq 0, \quad -x^2+6x-9<0$
  (d) $3x^2+x-1=0, \quad 3x^2+x-1 \geq 0, \quad 3x^2+x-1 \leq 0$

Optional Problems

1. For a given a, what is the minimum of the value of the expression x^2-ax+1? At what x that expression has a minimum value?

2. If $x+\frac{1}{x}=7$, find $x^2+\frac{1}{x^2}$; $x^3+\frac{1}{x^3}$.

3. Consider the sequence $x_1=1$, $x_2=\frac{x_1}{2}+\frac{1}{x_1}$, $x_3=\frac{x_2}{2}+\frac{1}{x_2}$ .... Compute the first several terms; does it seem that the sequence is increasing? Decreasing? Approaching some value? If so, can you guess this value? [Hint: solve equation $x=\frac{x}{2}+\frac{1}{x}$.]