MATH 7
ASSIGNMENT 5: GRAPH OF A QUADRATIC POLYNOMIAL: PARABOLA

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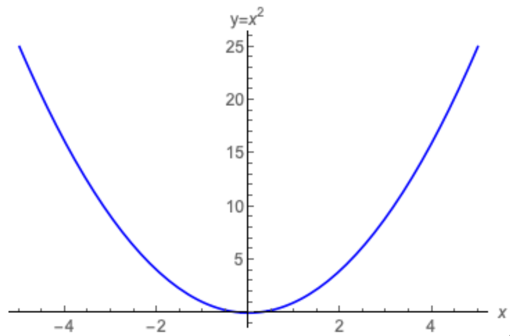
1. The Parabola

Today we will look at the graph of quadratic polynomials: the parabola, and how it helps understanding the roots, and quadratic inequalities.

The shape of the curve $y = ax^2 + bx + c$ is always that of a *parabola*. We will see later a more geometrical definition of the parabola, but for now it suffices to have an idea of how it looks like. For example, the curve $y = x^2$ is



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Other quadratic polynomials look similar. Let us use what we learned about them:

2. Quadratic Equations: Summary

- A **quadratic polynomial** is an expression of the form $p(x) = ax^2 + bx + c$.
- **Roots** of a quadratic polynomial are numbers such that $p(x) = 0$. If x_1, x_2 are roots, then $p(x) = a(x - x_1)(x - x_2)$.
- **Vietá formulas**: If x_1, x_2 are roots of $ax^2 + bx + c$, then

$$x_1 + x_2 = -\frac{b}{a} \tag{1}$$

$$x_1 x_2 = \frac{c}{a} \tag{2}$$

- **Completing the square**: we can rewrite

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a} = a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right) \tag{3}$$

where $D = b^2 - 4ac$.

From this, one gets the **quadratic formula**: if $D < 0$, there are no roots; if $D \geq 0$, then the roots are

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} \tag{4}$$

- From formula (3), we see that:
  - If $a > 0$, then the **smallest** possible value of $p(x)$ is $-\frac{D}{4a}$, which happens when $x = -\frac{b}{2a}$. In this case the graph is a parabola with branches going up.
  - If $a < 0$, then the **largest** possible value of $p(x)$ is $-\frac{D}{4a}$, which happens when $x = -\frac{b}{2a}$. In this case the graph is a parabola with branches going down.

– If $D < 0$, the parabola does not cross the x axis, while if $D > 0$ the parabola crosses the x axis at x_1 and x_2 given by (4)
– The point $(-b/2a, -D/4a)$ is called the *vertex* of the parabola

Homework

1. In each case, solve the equation, then the inequalities, and then sketch the graph of the parabola, pointing out the roots (if they exist) and the vertex, with their coordinates.
(a) $x^2 - 5x + 5 = 0$, $x^2 - 5x + 5 > 0$, $x^2 - 5x + 5 < 0$, sketch the graph $y = x^2 - 5x + 5$
(b) $x^2 - 5x - 14 = 0$, $x^2 - 5x - 14 > 0$, $x^2 - 5x - 14 < 0$, sketch the graph $y = x^2 - 5x - 14$
(c) $-x^2 + 11x - 28 = 0$, $-x^2 + 11x - 28 > 0$, $-x^2 + 11x - 28 < 0$, sketch the graph $y = -x^2 + 11x - 28$
(d) $-6x^2 - 19x + 7 = 0$, $-6x^2 - 19x + 7 > 0$, $-6x^2 - 19x + 7 < 0$, sketch the graph $y = -6x^2 - 19x + 7$
(e) $x^2 - x - 1 = 0$, $x^2 - x - 1 > 0$, $x^2 - x - 1 < 0$, sketch the graph $y = x^2 - x - 1$
(f) $-x^2 + 2x + 2 = 0$, $-x^2 + 2x + 2 > 0$, $-x^2 + 2x + 2 < 0$, sketch the graph $y = -x^2 + 2x + 2$
(g) $x^2 + 2x - 3 = 0$, $x^2 + 2x - 3 > 0$, $x^2 + 2x - 3 < 0$, sketch the graph $y = x^2 + 2x - 3$
(h) $x^2 + 2x + 3 = 0$, $x^2 + 2x + 3 \geq 0$, $x^2 + 2x + 3 < 0$, sketch the graph $y = x^2 + 2x + 3$
(i) $-x^2 + 6x - 9 = 0$, $-x^2 + 6x - 9 \geq 0$, $-x^2 + 6x - 9 < 0$, sketch the graph $y = -x^2 + 6x - 9$
(j) $3x^2 + x - 1 = 0$, $3x^2 + x - 1 \geq 0$, $3x^2 + x - 1 \leq 0$, sketch the graph $y = 3x^2 + x - 1$

2. For what values of a does the polynomial $x^2 + ax + 14$ have no roots? exactly one root? two roots?

3. The sum of reciprocals of two consecutive integers is 13/42. Find the integers. What are the consecutive integers for which the sum of their reciprocals is larger than 13/42? Less than 13/42?

4. Of all the rectangles with perimeter 4, which one has the largest area?
[**Hint:** if sides of the rectangle are a and b, then the area is $A = ab$, and the perimeter is $2a + 2b = 4$. Thus, $b = 2 - a$, so one can write A using only $a\ldots$]

5. What is the value of

$$x = \sqrt{2 + (\sqrt{2 + (\sqrt{2 + (\sqrt{2 + (\ldots}}}}$$

[**Hint:** Calculate x^2.]